$\Rightarrow \det(A^{\nu}, B^{\nu}) = (-1)^{1+2+\cdots+(n-1)} \det C = (-1)^{\frac{n(n-1)}{2}} \det C.$
Thus the index of v' at $P_\nu = \overset{sgn}{\gamma} \det C = (-1)^{n(n-1)/2}\, \mathrm{sgn} \det(A^{\nu}, B^{\nu})$
$= (-1)^{n(n-1)/2}\, \mathrm{sgn} \det(A_{P_\nu})$. ┘

We have, in this discussion, inverted the historical order of things. The Chern classes of complex vector bundles and the analogous Steifel-Whitney classes of real vector bundles were originally defined using obstruction theory; in terms of this definition, the classes were visibly the Poincare duals of degeneracy cycles. Chern then discovered the remarkable fact that these global topological invariants of a vector bundle could in fact be computed from the local hermitian differential geometric structure of the vector bundle; Chern's theorem has since been frequently adopted as a definition.

Some Remarks - Not indispensable - Concerning Chern Classes of Holomorphic Vector Bundles

Suppose that $E \to M$ is a holomorphic vector bundle with base space a complex manifold M. If we choose a hermitian connection as in Section 5 of Chapter 0, then the hermitian symmetry $\Theta + {}^t\bar{\Theta} = 0$ of the curvature matrix in a unitary frame implies the relations

$$C_p(\Theta) \text{ has type } (p,p), \quad C_p(\Theta) = \overline{C_p(\Theta)}$$

on the Chern forms.

Γ $\det\left(I + \frac{\sqrt{-1}}{2\pi} t\Theta\right) = 1 + tC_1 + t^2 C_2 + \cdots + t^p C_p + \cdots$