$$\bar{\partial}_k \left(\Sigma \varphi_{I\bar{J}} \, dz_I \wedge d\bar{z}_J \right) = \Sigma \frac{\partial \varphi_{I\bar{J}}}{\partial \bar{z}_k} \, dz_I \wedge d\bar{z}_J .$$

Note that ∂_k and $\bar{\partial}_k$ commute with e_ℓ, $\bar{e}_\ell$, ι_ℓ and $\bar{\iota}_\ell$ and with each other.

$\ulcorner$ $\partial_k e_\ell (f dz_I \wedge d\bar{z}_J) = \partial_k f \, dz_\ell \wedge dz_I \wedge d\bar{z}_J$

$$= \frac{\partial f}{\partial z_k} dz_\ell \wedge dz_I \wedge d\bar{z}_J = e_\ell \partial_k (f \, dz_I \wedge d\bar{z}_J) \lrcorner$$

$\ulcorner$ $\partial_k \bar{\partial}_{k'} (f \, dz_I \wedge d\bar{z}_J) = \partial_k \frac{\partial f}{\partial \bar{z}_{k'}} dz_I \wedge d\bar{z}_J$

$$= \frac{\partial^2 f}{\partial z_k \partial \bar{z}_{k'}} dz_I \wedge d\bar{z}_J = \frac{\partial^2 f}{\partial \bar{z}_{k'} \partial z_k} dz_I \wedge d\bar{z}_J = \bar{\partial}_{k'} \partial_k (f dz_I \wedge d\bar{z}_J) \lrcorner$$

Finally, we see that the adjoint of ∂_k is $-\bar{\partial}_k$; we have for $\varphi = \Sigma \varphi_{I\bar{J}} \, dz_I \wedge d\bar{z}_J$ any compactly supported form, L and M any multiindexes and ψ any C^∞ function,

$$\langle -\bar{\partial}_k \varphi, \psi \, dz_L \wedge d\bar{z}_M \rangle = \langle -\frac{\partial}{\partial \bar{z}_k} (\varphi_{L\bar{M}}) \, dz_L \wedge d\bar{z}_M , \psi \, dz_L \wedge d\bar{z}_M \rangle$$

$$= \int_{\mathbb{C}^n} - \frac{\partial}{\partial \bar{z}_k} \varphi_{L\bar{M}} \bar{\psi}$$

$$= \int_{\mathbb{C}^n} \varphi_{L\bar{M}} \frac{\partial}{\partial \bar{z}_k} (\bar{\psi}) \quad \text{(by integration by parts)}$$

$$= \int_{\mathbb{C}^n} \varphi_{L\bar{M}} \overline{\left(\frac{\partial \psi}{\partial z_k} \right)} = \langle \varphi_{L\bar{M}} \, dz_L \wedge d\bar{z}_M , \partial_k (\psi \cdot dz_L \wedge d\bar{z}_M) \rangle$$

$$= \langle \varphi , \partial_k (\psi \, dz_L \wedge d\bar{z}_M) \rangle$$

Likewise, the adjoint of $\bar{\partial}_k$ is $-\partial_k$.

We can express all of our operators on $A_c^{*,*}(\mathbb{C}^n)$ in terms of these elementary operators: clearly

$$\partial = \sum_k \partial_k e_k = \sum_k e_k \partial_k , \quad \bar{\partial} = \Sigma \bar{\partial}_k \bar{e}_k = \Sigma \bar{e}_k \bar{\partial}_k ,$$