$= f_k(g_1, \cdots g_k, z_{k+1}, \cdots z_n) = 0,$

we have at $(z_1^\circ, \cdots z_n^\circ)$.

$$\frac{\partial f_1}{\partial z_1}\cdot\frac{\partial g_1}{\partial z_{k+1}} + \frac{\partial f_1}{\partial z_2}\frac{\partial g_2}{\partial z_{k+1}} + \cdots + \frac{\partial f_1}{\partial z_k}\frac{\partial g_k}{\partial z_{k+1}} + \frac{\partial f_1}{\partial z_{k+1}} = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{\partial f_k}{\partial z_1}\frac{\partial g_1}{\partial z_{k+1}} + \frac{\partial f_k}{\partial z_2}\frac{\partial g_2}{\partial z_{k+1}} + \cdots + \frac{\partial f_k}{\partial z_k}\frac{\partial g_k}{\partial z_{k+1}} + \frac{\partial f_k}{\partial z_{k+1}} = 0.$$

$\Rightarrow \left\{ \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_1}{\partial z_2}, \cdots \frac{\partial f_1}{\partial z_{k+1}}\right), \cdots \left(\frac{\partial f_k}{\partial z_1}, \cdots \frac{\partial f_k}{\partial z_{k+1}}\right) \right\}$ is a set of vectors perpendicular to the vector $\left(\frac{\partial g_1}{\partial z_{k+1}}, \cdots \frac{\partial g_k}{\partial z_{k+1}}, 1\right)$.

By the assumption of k (k largest), the vector $\left(\frac{\partial f_i}{\partial z_1}, \cdots \frac{\partial f_i}{\partial z_{k+1}}\right)$ is a linear combination of $\left(\frac{\partial f_1}{\partial z_1}, \cdots \frac{\partial f_1}{\partial z_{k+1}}\right), \cdots \& \left(\frac{\partial f_k}{\partial z_1}, \cdots \frac{\partial f_k}{\partial z_{k+1}}\right)$.

$\Rightarrow \left(\frac{\partial f_i}{\partial z_1}, \cdots \frac{\partial f_i}{\partial z_{k+1}}\right)\cdot\left(\frac{\partial g_1}{\partial z_{k+1}}, \cdots \frac{\partial g_k}{\partial z_{k+1}}, 1\right) = \frac{\partial f_i(g_1, \cdots g_k, z_{k+1}, \cdots z_n)}{\partial z_{k+1}}$

$= 0.$

Similarly, $\frac{\partial f_i}{\partial z_{k+2}} = \cdots = \frac{\partial f_i}{\partial z_n} = 0$ at $(z_1^\circ, \cdots z_n^\circ)$.

$\Rightarrow \; df_i = 0$ at $(z_1^\circ, \cdots z_n^\circ)$

$$V'' = V' \cap U' \underset{G}{\overset{F}{\longleftrightarrow}} \mathbb{C}^{n-k}$$

$\Rightarrow$ f_i constat on V'' $\Rightarrow$ Since $\overline{V''} \supset V \cap U'$, and $f_i = 0$ on $V \cap U'$, $f_i = 0$ on V''.

$\Rightarrow \; V'' = V' \cap U' = \{ f_1 = \cdots = f_\ell = 0 \} = V \cap U'$ "

Here we proved that $V'' \subset V$. We already know that q near p $V \subset V''$ on a nbd of q, since all f_i's, $i = 1, \cdots k$ vanish on V.