

isomorphic. Note that any primitive element can be expressed as an element in  $P^{n-q-k, q}(M)$  or  $P^{n-q, q-k}(M)$ , and we can prove  $(v)\psi = 0$  for  $\psi \in P^{n-q, q-k}$  in the exactly same argument above. Recall that given any primitive group  $P^{x, y}(M)$  then  $x+y \leq n = \dim_{\mathbb{C}} M \Rightarrow d_1' = 0$  for all  $E_1^{p, q} \Rightarrow E_2^{p, q} = E_1^{p, q}$  and  $d_1' = d_2' \Rightarrow$  The same argument works, and  $d_2' = 0 \Rightarrow d_1' = d_2' = \dots = d_r' = \dots = 0 \Rightarrow$

We note that this proof has not used the full strength of duality. For example, the equality

$$\dim \text{Ext}^n(M; \mathcal{O}_{\mathbb{Z}}, \Omega^n) = \deg \mathbb{Z} = \dim H^0(M, \mathcal{O}_{\mathbb{Z}})$$

gives

$$\sum_p h^{p, p}(M) = \deg(\mathbb{Z}).$$

Since  $H^0(M, \mathcal{O}_{\mathbb{Z}}) \otimes \text{Ext}^n(M; \mathcal{O}_{\mathbb{Z}}, \Omega^n) \rightarrow \mathbb{C}$  is nondegenerate,  $\dim H^0(M, \mathcal{O}_{\mathbb{Z}}) = \dim \text{Ext}^n(M; \mathcal{O}_{\mathbb{Z}}, \Omega^n) = \dim H^0(M, \underline{\text{Ext}}_0^n(\mathcal{O}_{\mathbb{Z}}, \Omega^n))$  since  $\underline{\text{Ext}}_0^q(\mathcal{O}_{\mathbb{Z}}, \Omega^n) = 0$   $0 \leq q < n$ .

$$\Rightarrow \text{By P 707, } H^0(M, \underline{\text{Ext}}_0^n(\mathcal{O}_{\mathbb{Z}}, \Omega^n)) = \bigoplus_{p \in \mathbb{Z}} \underline{\text{Ext}}_{\mathcal{O}_p}^n(\mathcal{O}_{\mathbb{Z}, p}, \Omega_p^n)$$

$$= \bigoplus_{p \in \mathbb{Z}} \text{Ext}^n(\mathcal{O}_{\mathbb{Z}}/\mathcal{I}, \mathcal{O}) = \bigoplus_{p \in \mathbb{Z}} \mathcal{O}_{\mathbb{Z}}/\mathcal{I} \quad \text{by P 690.}$$

$\Rightarrow \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{Z}}/\mathcal{I} = \text{degree of a map } f: \mathbb{C}^n \rightarrow \mathbb{C}^n \text{ defined by } (f_1, \dots, f_n), \mathcal{I} = \langle f_1, \dots, f_n \rangle, (f_i = v_i, \dots, v_n = f_n \text{ in this case})$

$$\Rightarrow \dim_{\mathbb{C}} \left( \bigoplus_{p \in \mathbb{Z}} \mathcal{O}_{\mathbb{Z}}/\mathcal{I} \right) = \deg \mathbb{Z} = \sum m_{\nu} p_{\nu}, \quad \{p_{\nu}\} = \mathbb{Z}.$$

$m_{\nu}$  is the multiplicity, i.e.  $m_{\nu} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{Z}}}{\mathcal{I}_{\nu}}$ .

See P 667