$$\pi^* : H^0(M, \mathcal{J}_x(L^k)) \longrightarrow H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)).$$

$\mathrm{I\Gamma}$ $\sigma(x) = 0 \Rightarrow \tilde{\sigma}(p) = \pi^*\sigma(p) = \sigma(\pi(p)) = \sigma(x)$ for all $p \in E$. $\pi(p) = x \Rightarrow \tilde{\sigma}(p) = \pi^*\sigma(p) = \sigma(\pi(p)) = \sigma(x) = 0$.

$\Rightarrow \sigma(x) = 0 \Leftrightarrow \tilde{\sigma} \equiv 0$ on E.

$$\begin{tikzcd}
H^0(M, \mathcal{O}_x(L^k)) & H^0(\tilde{M}, \mathcal{O}_E(\tilde{L}^k)) \\
H^0(M, \mathcal{O}_M(L^k)) \arrow[u] \arrow[r, "\pi^*", "\cong"'] & H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k)) \arrow[u] \\
H^0(M, \mathcal{J}_x(L^k)) \arrow[u] \arrow[r, "\pi^*"] & H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)) \arrow[u]
\end{tikzcd}$$

Given $\tilde{\tau} \in H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E))$, $\tilde{\tau}|_E = 0$.

$\Rightarrow \exists$ a unique $\tau \in H^0(M, \mathcal{O}_M(L^k))$ s.t $\pi^*\tau = \tilde{\tau}$.

$\Rightarrow \tau(x) = 0$. $\Rightarrow \tau \in H^0(M, \mathcal{J}_x(L^k))$. $\Rightarrow$ The restriction of π^* is still isomorphic. $\lrcorner$

As before, we can identify

$$H^0(E, \mathcal{O}_E(\tilde{L}^k - E)) = L_x^k \otimes H^0(E, \mathcal{O}_E(-E)) \cong L_x^k \otimes T_x^{*\prime},$$

and the diagram

$$\begin{tikzcd}
H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^k - E)) \arrow[r, "r_E"] & H^0(E, \mathcal{O}_E(\tilde{L}^k - E)) \arrow[d, equal] \\
H^0(M, \mathcal{J}_x(L^k)) \arrow[u] \arrow[r, "d_x"] & T_x^{*\prime} \otimes L_x^k
\end{tikzcd}$$

commutes. Thus we must prove that r_E is surjective for $k \gg 0$.

$\mathrm{I\Gamma}$ $\tilde{L}^k - E|_E = \tilde{L}^k \otimes [-E]|_E = \tilde{L}_x^k \times E \otimes [-E]|_E$

and $\tilde{L}_x^k \times E \cong E \times \mathbb{C}$ trivial line bundle.