$$\Rightarrow \left\langle \sum \varphi_\xi e^{i\langle \xi, x\rangle}, \sum \|\xi\|^2 \eta_\xi e^{i\langle \xi, x\rangle} \right\rangle = \left\langle \sum \psi_\xi e^{i\langle \xi, x\rangle}, \sum \eta_\xi e^{i\langle \xi, x\rangle} \right\rangle$$

$$\sum_\xi \varphi_\xi \overline{\eta_\xi} \|\xi\|^2 = \sum_\xi \psi_\xi \overline{\eta_\xi}.$$

$\Rightarrow \quad \sum_\xi (\varphi_\xi \|\xi\|^2 - \psi_\xi)\overline{\eta_\xi} = 0 \quad$ for all η.

$\Rightarrow \quad \varphi_\xi \|\xi\|^2 = \psi_\xi \quad$ for all $\xi \quad \Rightarrow \quad$ in case $\xi = 0, \quad 0 = \psi_0$.

Thus ψ should be orthogonal to the weak solution space of $\Delta_d \varphi = 0$. (which we call the harmonic space) ⌟⌟

Now, assuming this to be the case,

$$\varphi = \sum_{\xi \neq 0} \frac{1}{\|\xi\|^2} \psi_\xi e^{i\langle \xi, x\rangle}$$ gives a formal

Fourier series solution to $\Delta_d \varphi = \psi$.

⌜⌜ $\Delta_d \varphi = \sum_{\xi \neq 0} \psi_\xi e^{i\langle \xi, x\rangle} = \psi \quad (\psi_0 = 0)$. ⌟⌟

Since clearly $\psi \in L^2(T) \Rightarrow \varphi \in L^2(T)$, it is a weak solution.

⌜⌜ $\|\varphi\|_0^2 = \sum_{\xi \neq 0} \left| \frac{\psi_\xi}{\|\xi\|^2} \right|^2 \leq \sum |\psi_\xi|^2 = \|\psi\|_0^2 < \infty$. ⌟⌟

In fact, we can say more: For $\psi \in L^2(T)$, if we define the Green's operator by

$$G(\psi) = \sum_{\xi \neq 0} \frac{1}{\|\xi\|^2} \psi_\xi e^{i\langle \xi, x\rangle},$$

then $G: H_s \longrightarrow H_{s+2}$ is a bounded linear operator.