But since $c_1([D+D']) = c_1([D]) + c_1([D']) = c_1([D]\otimes[D'])$,

$$\varphi^*[(d-n-1)H]|_V = (d-n-1)\,\varphi^*(H|_V) = (d-n-1)\, d_0\, H|_{V'}$$
$$= (d-n-1)\, H|_{V'} \quad \Rightarrow \quad d_0 = 1$$

$$\Rightarrow \varphi^*(H|_V) = H|_{V'}.$$

Remark: $V \xrightarrow{\varphi} V'$

$$
\begin{array}{ccc}
H^2(V', \mathbb{R}) & \xrightarrow{\varphi^*} & H^2(V, \mathbb{R}) \\
\in & & \in \\
c_1(L) & \longmapsto & c_1(\varphi^* L) = \varphi^*(c_1(L))
\end{array}
$$

$$c_1(L\otimes L') = c_1(L) + c_1(L') \longmapsto c_1(\varphi^*(L\otimes L'))$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $$
$$\qquad\qquad\qquad\qquad\qquad\qquad \varphi^*(c_1(L)) + \varphi^*(c_1(L))$$

In conclusion,

Two smooth hypersurfaces of dimension ≥ 3 and degree $d \neq n+1$ in $\mathbb{P}^n$ are isomorphic $\Leftrightarrow$ they are projectively isomorphic; or equivalently,

Any automorphism of a smooth hypersurface of dimension ≥ 3 and degree $d \neq n+1$ in $\mathbb{P}^n$ is induced by an automorphism of $\mathbb{P}^n$.

$\ulcorner$ ($\Rightarrow$) $V \subset \mathbb{P}^n$ $\quad f: V \longrightarrow V$ automorphism.
$\Rightarrow$ Since V is normal, and $\exists$ a bundle isomorph. $\tilde{f}$: $H|_V \longrightarrow H|_V$, $\Rightarrow$ f can be extended to an auto