$= 0$, since $\mathrm{Ord}_C(h) \geq \mathrm{Ord}_C(g)$. Q.E.D.

Of couse this particular lemma may be proved directly, but the method of using residues and local duality will work in a variety of circumstances.

As we set above, let $f(z,w) = z$. Assume $h(z,w) = z^p + b_1(w)z^{p-1} + \cdots + b_p(w)$. $g(z,w) = z^l + a_1(w)z^{l-1} + \cdots + a_l(w)$, without loss of generality.

$\Rightarrow$ Since $\mathrm{Ord}_{(z=0)} h \geq \mathrm{Ord}_{(z=0)} g$, $b_p(w) = w^m b_p'(w)$, $b_p'(0) \neq 0$. and $a_l(w) = w^k a_l'(w)$, $a_l'(0) \neq 0$, with $m \geq k$.

$h = z(\quad) + b_p(w) = z(\quad) + w^m b_p'(w) = z(\quad) + w^{m-k} b_p'(w) \frac{1}{a_l'(w)} (w^k a_l'(w)) = z(\quad) + (\quad)(g - z(\quad))$

$= z(\quad) + g(\quad) \in \{z, g\} = \{f, g\} = I$.

Thus we proved $h \in I$ directly, by using W.P.T.

Using the Max Noether theorem and this lemma, we shall prove the result on cubics encountered in Section 2:

Suppose that C, D, E are cubics in $\mathbb{P}^2$ and that each point $p \in C \cap D$ is a simple point on C. Suppose that for all but one such point $\mathrm{Ord}_C(E)_p \geq \mathrm{Ord}_C(D)_p$, and at the remaining one, say q, we have $\mathrm{Ord}_C(E)_q \geq \mathrm{Ord}_C(E)_q - 1$.

Then $\mathrm{Ord}_C(E)_q \geq \mathrm{Ord}_C(D)_q$. Briefly stated: any cubic E passing through eight of the nine points of $C \cap D$ passes through the remaining point also.