For each $i=1,2$, let $\mathbb{P}_i^{n-1}$ be an $(n-1)$-plane contained in the tangent plane $T_{p_i}(F)$ to F at p_i and not containing p_i; $\mathbb{P}_i^{n-1}$ then intersects F in a smooth quadric F_i, with $F \cap T_{p_i}(F)$ the cone through p_i over F_i.

$\ulcorner$ Note that $T_{p_i}(F) \cap F$ is a quadric in $T_{p_i}(F) = \mathbb{P}^n$ with the singular set $\{p_i\}$. $\Rightarrow$ Since $\mathbb{P}_i^{n-1}$ is an $(n-1)$-plane in the tangent plane $T_{p_i}(F) = \mathbb{P}^n$ and not containing p_i, the $T_{p_i}(F) \cap F$ quadric restricted to $\mathbb{P}_i^{n-1}$ is smooth.
$\Rightarrow$ $F \cap \mathbb{P}_i^{n-1}$ is smooth. and, as we saw on P734, P804 note, over $F \cap \mathbb{P}_i^{n-1} = F_i$, $F \cap T_{p_i}(F)$ is the cone through p_i. $\lrcorner$

Now the second condition on $\sigma_{n-k,n-k-1,\cdots}$ says that any $\Lambda \in \Sigma_{k,n} \cap \sigma_{n-k,n-k-1,\cdots}$ must meet the 3-plane $\overline{V_4}$ in a line.

$\ulcorner$ $\dim(\Lambda \cap V_4) \geq 2$. $\qquad \overline{V_4} = \mathbb{P}^3$.
$\dim(\Lambda \cap V_4) \geq 3$ is impossible, since otherwise $\Lambda \cap \overline{V_4} \supseteq \mathbb{P}^2$, which contradicts to the fact that no k-plane Λ lying on F can meet $\overline{V_\alpha}$ in a space of projective dimension $> (\alpha-2)/2$. If $\alpha = 4$, then $(4-2)/2 = 1$. $\quad \mathbb{P}^2 > \mathbb{P}^1$. $\lrcorner$

But $\overline{V_4}$ will meet $\mathbb{P}_i^{n-1}$ in a line, and F_i in a pair of points p_{i1}, p_{i2}; and writing any k-plane $\Lambda \subset F$ through p_i as