on M. Granted the general duality theorem, the proof of this stronger assertion runs about the same as the one we shall now give, which proceeds in two steps.

Step One in the proof. We denote by $\iota(v)$ the operation of contraction of a differential form with the vector field v. This operator was already encountered in the proof of the Bott residue theorem. See p429

If locally

$$\begin{cases} v = \sum_i v_i(z) \frac{\partial}{\partial z_i} \quad \text{and} \\ \varphi = \frac{1}{p!\,q!} \sum_{I,J} \varphi_{IJ}\, dz_I \wedge d\bar{z}_J \end{cases}$$

is a (p,q) form, then

$$\iota(v)\varphi = \frac{1}{(p-1)!\,q!} \sum_{I,J} \Big(\sum_{i \in I} \pm v_i \varphi_{IJ}\, dz_{I-\{i\}} \wedge d\bar{z}_J \Big).$$

$$\iota(v)\varphi = \frac{1}{p!\,q!} \sum_{I,J} \iota(v)\,(\varphi_{IJ}\, dz_I \wedge d\bar{z}_J)$$

$$= \frac{1}{p!\,q!} \sum_{I,J} \varphi_{IJ}\, \iota(v)(dz_I) \wedge d\bar{z}_J$$

$$=$$

i runs through I.

$$\frac{1}{p!\,q!} \sum_{I,J} \varphi_{IJ} \sum_{\alpha=1}^{p} (-1)^{\alpha-1} v_{i_\alpha}\, dz_{I-\{i_\alpha\}} \wedge d\bar{z}_J$$

See p428

$$= \frac{1}{p!\,q!} \sum_{I,J} \Big(\sum_{i\in I} \pm v_i\Big) \varphi_{IJ}\, dz_{I-\{i\}} \wedge d\bar{z}_J.$$