If σ_1 & σ_2 sections of $[H]$, and $\tau = a\sigma_1 + b\sigma_2$, then $\pi^*\tau = a\pi^*\sigma_1 + b\pi^*\sigma_2$. $\Rightarrow$ $\pi^* H^0(\mathbb{P}^1, \mathcal{O}(H))$ is the pencil we want.

$\lrcorner$

In fact, this argument tells us a bit more. Suppose that C' is a smooth point of V_C and that C' is simply tangent to C at a single point $p \in C$. If $L \subset W$ is a generic line through C' lying in the hyperplane $H_p \subset W$ of conics containing p, the corresponding pencil $\{C'_\lambda\}_{\lambda \in \mathbb{P}^1}$ will cut out on C a linear system of degree 4, with a base point. p as

$\ulcorner$ $H_p = \{$ all conics containing $p\} = \{ a_{00}X_0^2 + a_{11}X_1^2 + \cdots + a_{12}X_1X_2 = 0 \}$ $\quad L = \{C'_\lambda\}_{\lambda \in \mathbb{P}^1}$

$\Rightarrow \quad \bigcap C'_\lambda \ni p.$

$\lrcorner$

The corresponding map then expresses C as a 3-sheeted cover of $\mathbb{P}^1$, and so has only

$$\begin{aligned} b &= 2(g(C)-2) - 3(2g(\mathbb{P}^1)-2) \\ &= 4 \end{aligned}$$

branch points — i.e., the pencil $\{C'_\lambda\}$ can contain at most four conics tangent to C other than