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In general, $(V\cap W)^\perp = V^\perp + W^\perp$ & $(V+W)^\perp = V^\perp \cap W^\perp$ → orthogonal v.s to W
(Refer to P165. Lang. Linear Algebra)

$\Rightarrow$ If $\dim(\Lambda \cap V_{n-k+i-a_i}) = l \geq i$,

$$\dim(*\Lambda \cap V^*_{k-i+a_i}) = \dim(\Lambda^\perp \cap V^\perp_{n-k+i-a_i})$$
$$= -\dim(\Lambda^\perp + V^\perp_{n-k+i-a_i}) + \dim(\Lambda^\perp) + \dim(V^\perp_{n-k+i-a_i})$$
$$= -\dim((\Lambda \cap V_{n-k+i-a_i})^\perp) + \dim(\Lambda^\perp) + \dim(V^\perp_{n-k+i-a_i})$$
$$= -(n-l) + n-k + n-(n-k+i-a_i)$$
$$= -\cancel{n}+l+\cancel{n}-\cancel{k}+\cancel{n}-\cancel{n}+\cancel{k}-i+a_i = l-i+a_i \geq a_i.$$

$$\dim(\Lambda \cap V_{n-k+i-a_i}) = l \iff \dim(*\Lambda \cap V^*_{k-i+a_i}) = l-i+a_i$$

Proof of $(V\cap W)^\perp = V^\perp + W^\perp$. $(V+W)^\perp = V^\perp \cap W^\perp$

We will show that $(V+W)^\perp = V^\perp \cap W^\perp$

$x \in (V+W)^\perp \Rightarrow x \perp V+W \Rightarrow x \perp V$ and $x \perp W \Rightarrow x \in V^\perp \cap W^\perp \Rightarrow (V+W)^\perp \subset V^\perp \cap W^\perp$ --- ①

If $y \in V^\perp \cap W^\perp$, $\langle y, v+w\rangle = \langle y, v\rangle + \langle y, w\rangle = 0 \Rightarrow y \perp V+W \Rightarrow y \in (V+W)^\perp$.

$\Rightarrow V^\perp \cap W^\perp \subset (V+W)^\perp$. --- ②

By ① & ②, $(V+W)^\perp = V^\perp \cap W^\perp$.

$\Rightarrow ((V+W)^\perp)^\perp = (V^\perp \cap W^\perp)^\perp = V+W$ since $(V^\perp)^\perp = V$

Take $V^\perp = V$, $W^\perp = W$, we get

$$V^\perp + W^\perp = (V\cap W)^\perp.$$

Thus, for any a, the image $*\sigma_a \subset G(n-k, n)$ of the Schubert cycle $\sigma_a \subset G(k, n)$ is the Schubert cycle a^*, where a^* is defined to be the smallest nonincreasing sequence such that $a^*_{a_i} \geq i$ for all i.