get one of S_i's zero. $\Rightarrow$ This is the reason why we can not get $\sigma = 0$, which implies that we don't conclude ϕ is injective.

To show that the complete linear system $|kD|$ on C is cut out by hypersurfaces of degree, we are going to use the "some sort of" induction. We will show for $k=1$ and 2.

$k = 1$.
$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{O}(H)) & \xrightarrow{\phi} & H^0(C, \mathcal{O}(H)) \\ \rotatebox{90}{$\in$} & & \| \\ \sigma & \longmapsto & \sigma|_C. \quad \mathcal{O}(D) \end{array}$$

As we observed above, ϕ is injective and $\dim H^0(\mathbb{P}^n, \mathcal{O}(H)) = n+1$. $\Rightarrow$ Since $H^0(C, \mathcal{O}(D))$ has dim $n+1$, ϕ must be surjective.

$k=2$ Given $\tau \in H^0(\mathbb{P}^n, \mathcal{O}(2H))$, then we will get $\tau' = 0$ on D by adding some sections coming from hypersurfaces of degree 2. By the argument above, we have hypersurfaces of deg 2 where one of them does not contain any point of D.
$\Rightarrow$ If $\tau = 0$ on D, we can get $\tau' = \tau + \sum$ hyp. $= 0$ on D, see P643 note. Once we get τ', consider τ'/s_0 where $(s_0 = 0) = D$ and s_0 is obtained from a section in $H^0(\mathbb{P}^n, \mathcal{O}(H))$.
$\Rightarrow$ $\tau'/s_0 \in H^0(\mathbb{P}^n, \mathcal{O}(H))$. $\Rightarrow$ $\tau'/s_0 = \sum a_i \sigma_i$, where $\sigma_i \in H^0(\mathbb{P}^n, \mathcal{O}(H))$, since $\tau'/s_0 \in H^0(S, \mathcal{O}(H))$ and $H^0(S, \mathcal{O}(H)) =$ Image of ϕ.
$\Rightarrow$ $\tau' = \sum a_i \sigma_i \otimes s_0 \in H^0(\mathbb{P}^n, \mathcal{O}(2H))$
$\Rightarrow$ $\tau =$ hypersurfaces $+ \sum a_i \sigma_i \otimes s_0 \in H^0(\mathbb{P}^n, \mathcal{O}(2H))$ //