is contained in U_α and $\sum \eta_\alpha(\sigma|U_\alpha) = \sigma$ for $\sigma \in \mathcal{F}(W)$ are called fine, and the same argument shows that their higher cohomology groups vanish. $\lrcorner$

Thus, repeating the reasoning from the proof of de Rham's theorem,

$$H^q(M, \Omega^p(E)) \cong H^{p,q}_{\bar\partial}(E).$$

$\ulcorner$ $H^r(M, \mathcal{A}^{p,q}(E)) = 0$ for $r > 0$, all p,q.

$$\begin{aligned} H^q(M, \Omega^p(E)) &\cong H^{q-1}(M, \mathcal{Z}^{p,1}_{\bar\partial}(E)) \cong H^{q-2}(M, \mathcal{Z}^{p,2}_{\bar\partial}(E)) \\ &\cong H^1(M, \mathcal{Z}^{p,q-1}_{\bar\partial}(E)) \cong H^0(M, \mathcal{Z}^{p,q}_{\bar\partial}(E)) \big/ \bar\partial H^0(M, \mathcal{A}^{p,q-1}(E)) \\ &= \frac{\mathcal{Z}^{p,q}_{\bar\partial}(E) = Z^{p,q}_{\bar\partial}(E)}{\underset{\bar\partial A^{p,q-1}(E).}{\bar\partial \mathcal{A}^{p,q-1}(E) \,\shortparallel}} = H^{p,q}_{\bar\partial}(E) \end{aligned}$$

$\lrcorner$

Next we want to discuss harmonic theory in holomorphic vector bundles. Suppose we have metrics given on M and E; we have then induced metrics on all tangential tensor bundles of M tensored with E or E^*. In particular, if $\{\varphi_i\}$ is a local coframe for the metric on $T^{*\prime}_M$ and $\{e_\alpha\}$ a unitary frame for E, any section η of $A^{p,q}(E)$ can be written locally as

$$\eta(z) = \frac{1}{p!\,q!} \sum_{I,J,\alpha} \eta_{I,J,\alpha}(z)\, \varphi_I \wedge \bar\varphi_J \otimes e_\alpha;$$

for $\eta, \psi \in A^{p,q}(E)$,

$$(\eta(z), \psi(z)) = \frac{2^{p+q-n}}{p!\,q!} \sum_{I,J,\alpha} \eta_{I,J,\alpha}(z) \cdot \overline{\psi_{I,J,\alpha}(z)}.$$