1. $dP(\Theta_\alpha)=0$

2. The cohomology class $[P(\Theta_\alpha)] \in H^{2k}_{DR}(M)$ is independent of the connection chosen for E.

Proof. Writing $P(\Theta_\alpha) = \tilde{P}(\Theta_\alpha, \cdots \Theta_\alpha)$ for $\tilde{P}$ a polarization of P, by linearity

$$dP(\Theta_\alpha) = \sum \tilde{P}(\Theta_\alpha, \cdots d\Theta_\alpha, \cdots \Theta_\alpha).$$

For example, $\deg P = 1$

$$dP(\Theta_\alpha)(v_1, v_2, v_3) = v_1 P(\Theta_\alpha(v_2, v_3)) - v_2 P(\Theta_\alpha(v_1, v_3))$$
$$+ v_3 P(\Theta_\alpha(v_1, v_2)) - P(\Theta_\alpha([v_1, v_2], v_3)) + P(\Theta_\alpha([v_1, v_3], v_2))$$
$$- P(\Theta_\alpha([v_2, v_3], v_1)) = P(v_1(\Theta_\alpha(v_2, v_3))) - P(v_2 \Theta_\alpha(v_1, v_3))$$
$$+ P(v_3 \Theta_\alpha(v_1, v_2)) - \cdots = P(d\Theta_\alpha(v_1, v_2, v_3))$$
$$\Rightarrow dP(\Theta_\alpha) = P(d\Theta_\alpha).$$

For example $\tilde{P}(\omega, \eta) = \omega \wedge \eta$. ω, η even degree

$$d(\omega \wedge \eta) = d\omega \wedge \eta + \omega \wedge d\eta$$
$$= \tilde{p}(d\omega, \eta) + \tilde{p}(\omega, d\eta)$$

Since $\tilde{P}$ is linear in ω, η, $d(\omega \wedge \square) = d\omega \wedge \square$ $+ \omega \wedge d\square = \tilde{p}(d\omega, \ \ ) + \tilde{P}(\omega, \ \ )$.

$$\Rightarrow dP(\Theta_\alpha) = d\tilde{P}(\Theta_\alpha \cdots \Theta_\alpha) = \sum \tilde{P}(\Theta_\alpha, \cdots d\Theta_\alpha, \cdots \Theta_\alpha)$$

Now $\Theta_\alpha = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha$, so $d\Theta_\alpha = -d\theta_\alpha \wedge \theta_\alpha + \theta_\alpha \wedge d\theta_\alpha$. But $P(\Theta_\alpha)$ is invariant under change of frame for E, and as we saw for any $x_0 \in M$, we can find a frame