$U_\alpha \cap V \neq \phi$; we take the divisor (s) of the meromorphic section s to be given by

$$(s) = \sum_V \mathrm{ord}_V(s) \cdot V$$

With this convention, s is holomorphic $\Leftrightarrow$ (s) is effective.

Now, if $D \in \mathrm{Div}(M)$ is given by local data $f_\alpha \in \mathcal{M}(U_\alpha)$, then the functions f_α clearly give a meromorphic section s_f of $[D]$ with $(s_f) = D$.

$\ulcorner$ $D \in \mathrm{Div}(M)$ is given by local data $f_\alpha \in \mathcal{M}(U_\alpha)$.

$\Rightarrow$ $\dfrac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. $\qquad g_{\alpha\beta} = \dfrac{f_\alpha}{f_\beta}$.

$\Rightarrow$ $[D]$ is the bundle defined by $\{ g_{\alpha\beta}, \{U_\alpha\} \}$.

$\Rightarrow$ $(s_f) \equiv \sum \mathrm{ord}_V (f_\alpha) \cdot V \overset{?}{=} D$.

What is D? Local data f_α means that $f_\alpha = 0$

Let $D = \sum a_V V$. on, $V \cap U_\alpha$, $g_{V,\alpha} = 0$, $f_\alpha = g_{V,\alpha}^{\mathrm{ord}(f_\alpha)} \times \boxed{*}$

$= \sum \mathrm{ord}_V(f_\alpha) V. \quad \Leftarrow \quad a_V = \mathrm{ord}_V(f_\alpha)$. $\lrcorner$

Conversely, if L is given by trivializations φ_α with transition functions $g_{\alpha\beta}$ and s is any global meromorphic section of L, we see that

$$\frac{s_\alpha}{s_\beta} = g_{\alpha\beta},$$

i.e. $L = [(s)]$.

$\ulcorner$ s is any global meromorphic section of $L \Rightarrow$

$\exists$ an open cover $\{U_\alpha\}$ s.t $s_\alpha \in \mathcal{M}^*(U_\alpha)$. s.t

$g_{\alpha\beta} \doteq \dfrac{s_\alpha}{s_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta) \quad \Rightarrow \quad L = [(s)]$.