$\omega_{p-1,I} \in Z_{\bar\partial}^{n,n-p}(U_I)$, $\quad U_I = U_{i_1} \cap \cdots \cap U_{i_p} \cong (\Delta^*)^p \times \Delta^{n-p}$

$$= (U - D_{i_1}) \cap \cdots \cap (U - D_{i_p})$$

$$= (\bigcup D_{i_\ell})^c = U - \bigcup D_{i_\ell}$$

$\Rightarrow [\omega_{p-1,I}] \in H_{\bar\partial}^{n,n-p}(U_I) = H_{\bar\partial}^{n,n-p}((\Delta^*)^p \times \Delta^{n-p}) = 0$ by $p \geq 1$.

$\Rightarrow \quad 0 \longrightarrow Z_{\bar\partial}^{n,n-p-1}(U_I) \longrightarrow \mathcal{Q}^{n,n-p-1}(U_I) \longrightarrow Z_{\bar\partial}^{n,n-p}(U_I) \longrightarrow 0$

is exact. $\Rightarrow$ We don't need a refinement of $\underline{U}$, i.e., as P46 & P40, we have exact sequences of cochain groups

$$\begin{array}{ccccccccc} 0 \to & C^{p-1}(\underline{U}, Z_{\bar\partial}^{n,n-p-1}) & \longrightarrow & C^{p-1}(\underline{U}, \mathcal{Q}^{n,n-p-1}) & \xrightarrow{\bar\partial} & C^{p-1}(\underline{U}, Z_{\bar\partial}^{n,n-p}) & \to 0 \\ & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & \\ 0 \to & C^{p}(\underline{U}, Z_{\bar\partial}^{n,n-p-1}) & \longrightarrow & C^{p}(\underline{U}, \mathcal{Q}^{n,n-p-1}) & \xrightarrow{\bar\partial} & C^{p}(\underline{U}, Z_{\bar\partial}^{n,n-p}) & \to 0 \end{array}$$

$\lrcorner$

At the two extremes $p = n-1$ and $p = 0$ we obtain

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_P \omega = \sum_{i \in I} \int_{\Gamma_i} \omega_{0,i}$$

$$= \sum_{i \in I} \int_{P_i} \eta_\omega, \qquad \text{since } \eta_\omega = \omega_{0,i} \text{ in } U_i,$$

$$= \int_{\partial P_0} \eta_\omega, \quad \text{where } P_0 = \{z : |f_1(z)| \leq \epsilon, \ldots, |f_n(z)| \leq \epsilon\},$$

$$= \int_{S^{2n-1}} \eta_\omega. \qquad \text{Q.E.D.}$$

Pf

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_P \omega = \int_P \omega_{n-1} = \sum_{\#I=n} \int_{P_I} \omega_{n-1,I}, \qquad P_I = \{z : |f_i(z)| = \epsilon \text{ for all } i\}$$

$$= \sum_{\#I=1} \int_{P_I} \omega_{0,I} = \sum_{i \in \{1,2,\ldots,n\}} \int_{P_i} \omega_{0,i}$$

$P_i = \{z : |f_i(z)| = \epsilon, \&, \text{for } j \neq i, \ |f_j(z)| \leq \epsilon\}$