Date

$$H^0(S,\Omega^1) \longrightarrow H^0(S,\ \Omega^1(\Sigma P_\lambda)) \longrightarrow H^0(S,\oplus \mathbb{C}_{P_\lambda}) = \oplus\, \mathbb{C}_{P_\lambda}$$

$$\downarrow$$

$$H^1(S,\ \Omega^1) \cong \mathbb{C}$$

by Kodaira Vanishing theorem $\downarrow$

$$\Rightarrow H^1(S,\Omega^1(\Sigma P_\lambda)) = 0$$

$$\Rightarrow \quad \oplus\,\mathbb{C}_{P_\lambda} \big/ H^0(S,\Omega^1(\Sigma P_\lambda)) \cong \mathbb{C}.$$

But we have seen that the sum of the residues of any meromorphic 1-form on S is zero; so the image of $H^0(S,\Omega^1([\Sigma P_\lambda]))$ is contained in, hence equal to, the hyperplane $(\Sigma a_\lambda = 0) \subset \oplus\, \mathbb{C}_{P_\lambda}$ Q.E.D.

See P 222. Choose a point $(a_\lambda) \in \oplus\,\mathbb{C}_{P_\lambda}$ s.t $\Sigma a_\lambda = 0$. $\Rightarrow \exists$ a section $\sigma \in H^0(S,\Omega^1([\Sigma P_\lambda])$ s.t $\sigma(P_\lambda) = a_\lambda$. Let s_0 be a section of $[\Sigma P_\lambda]$ s.t $(s_0 = 0) = \Sigma P_\lambda$.

$\Rightarrow$ Consider σ/s_0. $\Rightarrow$ σ/s_0 is a meromorphic 1-form on S whose residues are $a_\lambda \cdot b_\lambda$ where b_λ = constant coming from s_0. which are not determined.

Thus we need to find another way.

Let $\{(a_1, a_2, \ldots a_\lambda, \ldots a_\ell) \in \oplus\,\mathbb{C}_{P_\lambda} \mid \Sigma a_\lambda = 0\}$

$$\sigma \in H^0(S,\Omega^1([\Sigma P_\lambda])) \longrightarrow \oplus\,\mathbb{C}_{P_\lambda}$$

$$\updownarrow \qquad\qquad \updownarrow$$

$\sigma/s_0 \in$ Set of all meromorphic 1-forms on S with a simple pole at each P_λ