Note that at each point $q \in f(M)$ the Lelong number
$\Theta(S, q) \geq 1$.
This is true at points $q = f(p)$, where $p \in M^*$ and f has maximum rank, since then one piece of $f(M)$ passing through q will be a complex manifold, and it is therefore true on all of S by semicontinuity.

$$\Theta(S, q) = \frac{1}{\pi^{n-p}} \lim_{r \to 0} \Theta(S, q, r)$$

$$= \frac{1}{\pi^{n-p}} \lim_{r \to 0} \frac{1}{r^{2n-2p}} S(\chi(r)\, \omega^{n-p})$$

Suppose $U \xrightarrow{f} \mathbb{C}^N$ has the Jacobian of f rank k for all points in U and $\left|\left(\frac{\partial f_i}{\partial z_j}\right)\right| \neq 0 \quad 1 \leq i, j \leq k$.

$\Rightarrow$ By the argument above, we have an open set W in U s.t $W \ni 0$,

$$f(W) = f(W \cap \mathbb{C}^k).$$

$\Rightarrow$ Locally, $f : \mathbb{C}^k \longrightarrow f(U)$ is biholomorphic, which implies that, locally, $f(U)$ is a k-dimensional complex manifold.

Let $q = f(p)$, where $p \in M^*$ and f has maximum rank. $\Rightarrow \exists$ open set $U \ni p$ s.t f has maximum rank on U. $\Rightarrow$ By the previous argument, $\exists\, U' \ni p$ s.t $f(U')$ is biholomorphic to $\mathbb{C}^k$. Restrict S to $f(U') \subset \Delta$.