Note that since ω is real, Q defines a real bilinear form on $H^{n-k}(M, \mathbb{R})$. By consideration of type, we see that

$$Q(H^{p,q}, H^{p',q'}) = 0 \quad \text{unless } p = q',\ q = p'$$

Γ $p+q = n-k \qquad p'+q' = n-k$
$p+p' = q+q' = n-k \qquad \Rightarrow \quad p' = q, \quad q = p'$ 」

The Hodge–Riemann bilinear relations assert that for $\xi \in P^{p,q}(M)$, a primitive class and $k = p+q$,

$$(i)^{p-q}\,(-1)^{(n-k)(n-k-1)/2}\quad Q(\xi, \bar{\xi}) > 0.$$

In the case $p+q$ even, this is the same as saying that on the real vector space

$$(P^{p,q} \oplus P^{q,p}) \cap H^{p+q}(M, \mathbb{R}) = \{\xi + \bar{\xi};\ \xi \in P^{p,q}(M)\} \subset H^{p+q}(M),$$

the quadratic form $(i)^{p-q}(-1)^{(n-k)(n-k-1)/2}\,Q$ is positive definite; in the case $p+q$ odd,

the bilinear relations tell us at least that Q is a nondegenerate skew-symmetric form on $P^{p+q}(M)$.

Γ ① $(P^{p,q} \oplus P^{q,p}) \cap H^{p+q}(M, \mathbb{R}) \ni \eta = \eta_{p,q} + \eta_{q,p}$ s.t
$\bar{\eta} = \eta. \quad \Rightarrow \quad \overline{\eta_{p,q}} + \overline{\eta_{q,p}} = \eta_{p,q} + \eta_{q,p}$
By comparing types, $\quad \overline{\eta_{p,q}} = \eta_{q,p} \quad \& \quad \eta_{p,q} = \overline{\eta_{q,p}}$

$\Rightarrow \eta = \eta_{p,q} + \overline{\eta_{p,q}} \in \{\xi + \bar{\xi};\ \xi \in P^{p,q}(M)\}$
$\Rightarrow (P^{p,q} \oplus P^{q,p}) \cap H^{p+q}(M, \mathbb{R}) \subset \{\xi + \bar{\xi};\ \xi \in P^{p,q}(M)\}$

Given $\xi + \bar{\xi}$, where $\xi \in P^{p,q}(M)$, since $H^{p,q} = \overline{H^{q,p}}$,
$\xi + \bar{\xi} \in P^{p,q} \oplus P^{q,q}$ and $\xi + \bar{\xi}$ real in $H^{p+q}(M, \mathbb{C})$
$\Rightarrow \xi + \bar{\xi} \in H^{p+q}(M, \mathbb{R}), \quad \Rightarrow \xi + \bar{\xi} \in (P^{p,q} \oplus P^{q,q}) \cap H^{p+q}(M, \mathbb{R}).$