Gauss - Bonnet Theorem I

$$C_r(S) = (-1)^r \cdot \sigma^*_{1,\cdots,1}.$$

Proof. By our computation of the intersection pairing in $H_*(G(k,n), \mathbb{Z})$, we must show that for any Schubert cycle σ_a of dimension r,

$$C_r(S)(\sigma_a) = (-1)^r \#(\sigma_{1,\cdots,1}\cdot\sigma_a)$$

$$= \begin{cases} (-1)^r, & \text{if } a = n-k, \cdots n-k, n-k-1, \cdots n-k-1. \\ 0, & \text{otherwise.} \end{cases}$$

$\ulcorner$ $C_r(S) \in H^{2r}_{DR}(G(k,n))$ $\quad$ $\dim \sigma_a = 2k(n-k) - 2\Sigma a_i$
$= 2r$

$\operatorname{codim} \sigma_{1,1,\cdots,1} = -(2k(n-k) - 2r) + 2k(n-k) \Rightarrow \dim \sigma_{1,\cdots,1} = \begin{matrix} 2k(n-k) \\ -2r \end{matrix}$

$\Rightarrow \quad 1+1+\cdots+1 = r$

$\Rightarrow$ If $\sigma_{1,\cdots,1} = \sigma_{\underbrace{1,1,\cdots,1}_{r},\underbrace{0\cdots0}_{k-r}}$ & $\sigma_{n-k,n-k,\cdots n-k,n-k-1,\cdots n-k-1}$,

$\Rightarrow \#(\sigma_{1,\cdots,1}\cdot\sigma_{n-k,\cdots n-k,n-k-1,\cdots n-k-1}) = 1.$

$\dim \sigma^*_{1,\cdots,1} = 2r$

$$\sigma^*_{1\cdots1}(\sigma_a) = \#(\sigma_{1,\cdots,1}\cdot\sigma_a) = \begin{cases} 1, & \text{if } a = n-k, \cdots n-k, n-k-1, \cdots \\ 0 & \text{otherwise} \end{cases}$$

$\lrcorner$

We first note that if $\sigma_a(V)$ is any Schubert cycle of dimension r, and $a \neq n-k, \cdots n-k, n-k+0-1, \cdots n-k-1$,