tells us that there does indeed exist a meromorphic function $F$ on $S$ with a double pole at $p$, holomorphic elsewhere.

⌈ $H^1(S, \mathcal{O}(p)) = H^1(S, \Omega^1([p]\otimes K_S^*)) = 0$ since
$\deg([p]\otimes K_S^*) = \deg[p] + \underset{0}{\underset{\|}{\deg K_S^*}} = 1 > 0 \Leftrightarrow [p]\otimes K_S^*$ is positive.

$$0 \longrightarrow \underset{\mathcal{O}(p)}{\underset{\|}{\mathcal{O}(2p-p)}} \longrightarrow \mathcal{O}(2p) \longrightarrow \underset{\mathbb{C}_p}{\underset{\|}{\mathcal{O}_p(p)}} \longrightarrow 0$$

$$\Rightarrow H^0(S, \mathcal{O}(p)) \longrightarrow H^0(S, \mathcal{O}(2p)) \longrightarrow \underset{\mathbb{C}_p}{\underset{\|}{H^0(S, \mathbb{C}_p)}} \longrightarrow \underset{H^1(S,\mathcal{O}(p))}{\underset{\|}{0}}$$

$\Rightarrow 1 \in \mathbb{C}_p \Rightarrow \exists$ a section $\sigma \in H^0(S, \mathcal{O}(2p))$ s.t. $\sigma(p) = 1$. $\Rightarrow \sigma \otimes s_0^{-1} = \sigma/s_0$ is a meromorphic function $F$ on $S$ with a double pole at $p$, holomorphic elsewhere, where $(s_0 = 0) = 2p$. ⌋

Next $h^0(S, \Omega^1) = g(S) = 1$, so $S$ has a nonzero holomorphic 1-form $\omega$; since $\deg K_S = \deg(\omega) = 0$, $\omega$ must be everywhere nonzero.

⌈ $h^0(S, \Omega^1) = \dim H^0(S, \Omega^1) = 1 \Rightarrow 0 \neq \omega \in H^0(S, \Omega^1)$.
$\deg K_S = 0$. If $\omega = 0$ at some point $p \in S$, then $\deg K_S = \deg(\omega = 0) \geq 1$. $\Rightarrow$ Contradiction. $\Rightarrow$ $\omega$ must be everywhere nonzero. ⌋

Consider the meromorphic form $F\cdot\omega$; it is holomorphic on $S - \{p\}$, and by the residue theorem

$$\mathrm{Res}_p(F\cdot\omega) = 0.$$

Consequently if $z$ is any local coordinat around $p$, after