The Bott Residue Formula

We ask now whether there exist refinements of the Gauss - Bonnet formula for holomorphic vector bundles on complex manifolds. The answer, in general, is no, for the reason that a zero of a section σ of a holomorphic vector bundle E on a complex manifold carries no nonobvious local structure: since we can choose a frame $e = (e_1, \cdots e_k)$ for E and a local holomorphic coordinate system $z = (z_1, \cdots z_n)$ for M independently, the local expansion

$$\sigma(z) = \sum b_{ij} z_i \cdot e_j + \sum b_{ijl} z_i z_j \cdot e_l + \cdots$$

for σ can be given virtually arbitrary form. The exception to this occurs when E is a holomorphic tensor bundle, e.g. when $E = T'(M)$ is the holomorphic tangent bundle of M: in this case a local coordinate system (z_i) determines naturally a frame $\{\partial/\partial z_i\}$ for $T'(M)$. Thus, in the nbd of a zero of the holomorphic vector field v, we set $A_p = (a_{ij})$, where

$$v(z) = \sum a_{ij} z_i \frac{\partial}{\partial z_j} + [2].$$

If $w = f(z)$ is any other coordinate system around $z=0$ and we let $A_p' = (a_{ij}')$ be given by

$$v(z) = \sum a_{ij}' w_i \cdot \frac{\partial}{\partial w_j} + [2],$$

then for $g = (g_{ij}) = J(f)$ the Jacobian of the change of coordinates,