Date 40f

$$\mathrm{Sym}^d(\mathbb{C}^{n+1*}) \xrightarrow{\phi} \{ F : \overbrace{\mathbb{C}^{n+1} \otimes \cdots \otimes \mathbb{C}^{n+1}}^{d} \longrightarrow \mathbb{C}, \text{ symmetric} \}$$

$$\langle e_1^* \otimes \cdots \otimes e_d^* \rangle \longmapsto \frac{1}{r_1! \cdots r_d!\, d!} \sum e_{\sigma(1)}^* \otimes \cdots \otimes e_{\sigma(d)}^*$$

where $r = \#$

$$\sum F(e_{i_1} \cdots e_{i_d})\, d!\, E_{I_d} \xleftarrow{\psi} F$$

where $\langle e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* \rangle = E_{I_d}$, $I_d = \{ i_1, i_2 \cdots i_d \}$

$\Rightarrow$ By construction, $\phi \circ \psi = \text{identity}$. $\psi \circ \phi = id$.

$\Rightarrow$ $\mathrm{Sym}^d(\mathbb{C}^{n+1*}) = \{ F : \mathbb{C}^{n+1} \otimes \cdots \otimes \mathbb{C}^{n+1} \longrightarrow \mathbb{C}, \text{ symmetric} \}$

And we can show the above isomorphism by using the universal mapping property of symmetric tensor algebra.

$$F(e_1, e_1, e_2)\ 3! \frac{1}{3!} \{ 2 e_1^* \otimes e_1^* \otimes e_2^* + 2 e_2^* \otimes e_1^* \otimes e_1^* + 2 e_1^* \otimes e_2^* \otimes e_1^* \} (e_1, e_1, e_2)$$

Correction. $= F(e_1, e_1, e_2) \cdot 2.$

$\langle e_{i_1}^* \otimes \cdots \otimes e_{i_d}^* \rangle$

$$= \frac{1}{d!\, r_1! \cdots r_d!} \sum_{\sigma \in S_d} e_{i_{\sigma(1)}}^* \otimes e_{i_{\sigma(2)}}^* \otimes \cdots \otimes e_{i_{\sigma(d)}}^*$$

where $\#\{ i_1 = i_l \} = r_1$

$\#\{ i_2 = i_l \} = r_2$

$\#\{ i_d = i_l \} = r_d$.

once, some $i_l = i_j$, if it appears again

then $r_j = 0$.