But at $t=0$

$$\frac{\partial}{\partial t}(\psi + t\bar\partial\eta,\ \psi + t\bar\partial\eta) = 2\,\mathrm{Re}(\psi, \bar\partial\eta)$$

and $\quad \frac{\partial}{\partial t}(\psi + t\bar\partial(i\eta),\ \psi + t\bar\partial(i\eta)) = 2\,\mathrm{Im}(\psi, \bar\partial\eta)$

$$\Rightarrow \quad \mathrm{Re}(\psi, \bar\partial\eta) = 0 \ \&\ \mathrm{Im}(\psi, \bar\partial\eta) = 0 \Rightarrow \underset{(\bar\partial^*\psi,\ \eta)}{\underset{\parallel}{(\psi, \bar\partial\eta)}} = 0$$

for all $\eta \in A^{p,q-1}(M) \Rightarrow \bar\partial^*\psi = 0$. Q.E.D.

So, at least formally, the ~~Deo~~ Dolbeault cohomology group $H^{p,q}_{\bar\partial}(M) = Z^{p,q}_{\bar\partial}(M)/\bar\partial A^{p,q-1}(M)$ is represented exactly by the solutions of the two first-order equations

$$\bar\partial\psi = 0, \quad \bar\partial^*\psi = 0.$$

These two may be replaced by the single-order equation

$$\Delta_{\bar\partial}\psi = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\psi = 0;$$

clearly $\bar\partial\psi = \bar\partial^*\psi = 0 \Rightarrow \Delta\psi = 0$, and conversely,

$$(\Delta_{\bar\partial}\psi, \psi) = \langle \bar\partial\bar\partial^*\psi, \psi\rangle + \langle \bar\partial^*\bar\partial\psi, \psi\rangle = \|\bar\partial^*\psi\|^2 + \|\bar\partial\psi\|^2$$

so $\Delta_{\bar\partial}\psi = 0 \Rightarrow \bar\partial\psi = 0 = \bar\partial^*\psi$.

The operator

$$\Delta_{\bar\partial} : A^{p,q}(M) \longrightarrow A^{p,q}(M)$$ is called the $\bar\partial$-Laplacian

or simply the Laplacian (written Δ) if no ambiguity is likely.

Differential forms satisfying the Laplacian equation $\Delta\psi = 0$ are called harmonic forms; the space of harmonic forms of type (p,q) is denoted $\mathcal{H}^{p,q}(M)$ and called the harmonic space. What the above formal argument suggests