At the other extreme, we have the

Lemma (Poincare-Lelong Equation). If the holomorphic function $f \in \mathcal{O}(M)$ has divisor the analytic hypersurface Z, then the equation of currents

$$T_Z = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |f|$$

is valid.

Proof. Around a smooth point we may choose coordinates $(z_1, z_2, \cdots z_n)$ such that $f(z) = z_n$. Then

$$\frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |f| = \bar{\partial} \left(\frac{1}{2\pi\sqrt{-1}} \partial \log f \right)$$

$$= \bar{\partial} \left(\frac{1}{2\pi\sqrt{-1}} \frac{dz_n}{z_n} \right)$$

$$= T_{\{z_n = 0\}}$$

by an obvious extension of the 1-variable Cauchy formula (in distributional form)

$$\bar{\partial} \left(\frac{1}{2\pi\sqrt{-1}} \frac{dz}{z} \right) = \delta_{\{0\}}$$

allowing dependence on parameters.

Γ $\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |f|^2 = \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |f| = + \frac{1}{2\pi\sqrt{-1}} \bar{\partial} \partial \log |f|^2$

$= + \frac{1}{2\pi\sqrt{-1}} \bar{\partial} \partial (\log f + \log \bar{f}) = + \frac{1}{2\pi\sqrt{-1}} \bar{\partial} \partial \log f$

$= \frac{1}{2\pi\sqrt{-1}} \bar{\partial} \partial \log z_n = \bar{\partial} \left(\frac{1}{2\pi\sqrt{-1}} \partial \log z_n \right) = \bar{\partial} \left(\frac{1}{2\pi\sqrt{-1}} \frac{dz_n}{z_n} \right)$

$= T_{\{z_n = 0\}}$ by P370. $\lrcorner$