

If locally $I = \{f_1, \dots, f_m\}$ is generated by holomorphic functions f_1, \dots, f_m , then the support of \mathcal{O}/I is defined as

$$\begin{aligned}\mathbb{Z} &= \text{supp}(\mathcal{O}/I) \\ &= \{z \in M : I_z \neq \mathcal{O}_z\} \\ &= \{z \in M : f_1(z) = \dots = f_m(z) = 0\}.\end{aligned}$$

$$\begin{aligned}\text{If } I_z \neq \mathcal{O}_z &\Rightarrow z \in \{z : f_i(z) = 0\}. \quad z \notin \{z : f_i(z) = 0\}. \\ \Rightarrow I_z = \mathcal{O}_z &\text{ since one of } f_i \text{ is a unit.} \quad \square\end{aligned}$$

As a point set \mathbb{Z} is an analytic variety. However, this should be refined, and the pair $(\mathbb{Z}, \mathcal{O}_{\mathbb{Z}})$ should be thought of as a space whose support is an analytic variety but whose structure sheaf $\mathcal{O}_{\mathbb{Z}} = \mathcal{O}/I$ is a sheaf of rings possibly with nilpotent elements. These objects are called Schemes.

$$\begin{aligned}\text{For example, } M &= \mathbb{C}^3. \quad I = \{z_3\}. \\ \Rightarrow \mathbb{Z} &= \text{supp}(\mathcal{O}/I) = \{z \in \mathbb{C}^3 : I_z \neq \mathcal{O}_z\} = \{z \in \mathbb{C}^3 : \\ z_3 &= 0\} = \mathbb{C}^2 \times \{0\} = \mathbb{C}^2. \quad \Rightarrow \mathcal{O}_{\mathbb{Z}} \stackrel{\textcircled{2}}{=} \mathcal{O}(\mathbb{C}^2)\end{aligned}$$

$$\begin{aligned}\text{Define } G: \mathcal{O}(\mathbb{C}^2) &\longrightarrow \mathcal{O}_{\mathbb{Z}} \text{ by} \\ \psi &\longmapsto \psi + \{z_3\}\end{aligned}$$

$$\begin{aligned}\text{For any } f \in \mathcal{O}(\mathbb{C}^3), \text{ at a point } z^0 = (z_1^0, z_2^0, z_3^0), \\ f &= z_3 f'(z_1, z_2) + f''(z_1, z_2). \Rightarrow G \text{ is isomorphic} \\ \Rightarrow \mathcal{O}(\mathbb{C}^2) &\cong \mathcal{O}_{\mathbb{Z}}.\end{aligned}$$