$$\Delta^p \xrightarrow{f_i} M \xrightarrow{\quad f \quad} N.$$

$$\begin{tikzcd}
\int\varphi = [\varphi] \in H^p_{sing}(N,\mathbb{R}) \arrow[r, "f^*"] \arrow[dr, phantom, "?"] & H^p_{sing}(M,\mathbb{R}) \ni \int f^*\varphi. \\
\varphi \in H^p_{DR}(N) \arrow[u] \arrow[u, leftrightarrow, "\cong"'] \arrow[r, "f^*"] & H^p_{DR}(M) \arrow[u, leftrightarrow, "\cong"'] \ni f^*\varphi.
\end{tikzcd}
\qquad \sigma \in H_p(M,\mathbb{R}) = \sum a_i f_i$$

$$\langle f^*\varphi, \sigma\rangle \underset{}{=} \int_\sigma f^*\varphi = \sum_i a_i \int_\Delta f_i^*(f^*\varphi) = \sum_i a_i \int_\Delta (f\circ f_i)^*\varphi$$

, since $\quad f_*\sigma = \sum a_i \, f\circ f_i$,

$$= \cancel{\sum_i} \int_{\sum a_i f\circ f_i} \varphi = \langle \varphi, \sum a_i f\circ f_i\rangle = \langle \varphi, f_*\sigma\rangle$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad \langle f^*\varphi, \sigma\rangle$$

$\Rightarrow \langle f^*\varphi, \sigma\rangle = \langle \varphi, f_*\sigma\rangle$

$\Rightarrow [f^*\varphi](\sigma) = [\varphi](f_*\sigma)$ where $[f^*\varphi] \in H^p_{sing}(M,\mathbb{R})$ representing $f^*\varphi$.

$\Rightarrow [f^*\varphi](\sigma) = f^*[\varphi](\sigma) \quad \Rightarrow [f^*\varphi] = f^*[\varphi]$.

$\Rightarrow$ The diagram is commutative $\Rightarrow$ The de Rham map is functorial.

The Dolbeault Theorem.

M complex manifold, $\Rightarrow H^q(M, \Omega^p) = H^{p,q}_{\bar\partial}(M)$.

pf). By the $\bar\partial$ - Poincare lemma, the sequence of sheaves

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \xrightarrow{\bar\partial} \mathcal{A}^{p,3} \longrightarrow$$

on M is exact.