$$\int_{\pi^{-1}(U_1)} \pi^*\rho_1 \cdot \omega \wedge \pi^*\eta = \int_{V_{1,1}} \pi^*\rho_1 \cdot \omega \wedge \pi^*\eta + \int_{V_{1,2}} \pi^*\rho_1 \cdot \omega \wedge \pi^*\eta + \cdots$$

$$+ \int_{V_{1,m}} \pi^*\rho_1 \cdot \omega \wedge \pi^*\eta,$$ where $V_{1,i} \xrightarrow{\pi} U_1$ homeomorph (diff)

Let $\psi : U_1 \longrightarrow R^n$ be a coordinate diffeomorphism.

$$\Rightarrow \int_{V_{1,i}} \pi^*\rho_1 \cdot \omega \wedge \pi^*\eta \overset{\text{def}}{\equiv} \int_{R^n} (\pi^{-1} \circ \psi^{-1})^* (\pi^*\rho \cdot \omega \wedge \pi^*\eta)$$

$$\begin{tikzcd} V_{1,i} \arrow[r, shift left, "\pi"] & U_1 \arrow[l, shift left, "\pi^{-1}"] \arrow[r, shift left, "\psi"] & R^n \arrow[l, shift left, "\psi^{-1}"] \end{tikzcd} \qquad = \int_{R^n} (\psi^{-1})^*(\rho_1 \cdot ((\pi^{-1})^*\omega \wedge \eta))$$

$$(\pi^{-1} \circ \psi^{-1})^* \pi^*\rho_1 = (\pi \circ \pi^{-1} \circ \psi^{-1})^* \rho_1 = (\psi^{-1})^*\rho_1$$

$$\begin{aligned} &(\pi^{-1} \circ \psi^{-1})^* (\omega \wedge \pi^*\eta) \\ &= (\pi^{-1} \circ \psi^{-1})^*\omega \wedge (\pi^{-1} \circ \psi^{-1})^* \pi^*\eta \\ &= (\pi^{-1} \circ \psi^{-1})^*\omega \wedge (\psi^{-1})^*\eta \\ &= (\psi^{-1})^*((\pi^{-1})^*\omega) \wedge (\psi^{-1})^*\eta \\ &= (\psi^{-1})^*(\pi^{-1*}\omega \wedge \eta) \end{aligned}$$

$$= \int_{U_1} \rho_1 \cdot (\pi^{-1}|_{V_{1,i}})^*\omega \wedge \eta$$

$$\Rightarrow \sum_i \int_{U_1} \rho_1 (\pi^{-1}|_{V_{1,i}})^*\omega \wedge \eta = \int_{U_1} \rho_1 \, \omega' \wedge \eta$$

Similarly, we get $\int_{\pi^{-1}(U_2)} \pi^*\rho_2 \cdot \omega \wedge \pi^*\eta = \int_{U_2} \rho_2 \, \omega' \wedge \eta$

$$\Rightarrow \int_{\tilde{M}} \omega \wedge \pi^*\eta = \int_{\pi^{-1}(U_1) \cup \pi^{-1}(U_2)} (\pi^*\rho_1 + \pi^*\rho_2) \cdot \omega \wedge \pi^*\eta =$$