$\sigma(q) \neq 0$ by the seperation of the linear system,
$[(\sigma_1(p), \sigma_2(p), \sigma_3(p))] \neq [(\sigma_1(q), \sigma_2(q), \sigma_3(q))]$, otherwise
$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\iota_L(p)} \quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{\iota_L(q)}$.

$\sigma_1(p) = \lambda\sigma_1(q)$, $\sigma_2(p) = \lambda\sigma_2(q)$, $\sigma_3(p) = \lambda\sigma_3(q) \Rightarrow$
if we let $\sigma = a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, then $\sigma(p) = 0 = \sigma(q)$,
which is impossible. $\Rightarrow$ $\iota_L : M \longrightarrow \mathbb{P}^2$ is one to one &
holomorphic $\Rightarrow$ By the propositon on P19, locally $\iota_L$ is
biholomorphic, and surjective. Hence $\iota_L$ is isomophism. ⏎

We have now shown that every smooth cubic surface $S \subset \mathbb{P}^3$ is of the form $\widetilde{\mathbb{P}}^2_{p_1,\dots p_6}$. Suppose that three of points $p_i$ lay on a line $L \subset \mathbb{P}^2$. Then the proper transform $\widetilde{L}$ of $L$ in $S$ would be a smooth rational curve of self-intersection $1\cdot 1 - 3 = -2$, and by the adjunction formula,

$$0 = \pi(\widetilde{L}) = \frac{\widetilde{L}\cdot\widetilde{L} + K_S\cdot\widetilde{L}}{2} + 1,$$

i.e., $K_S \cdot \widetilde{L} = 0$.

$\Gamma$ $\widetilde{L}$ is a smooth rational curve in $\widetilde{\mathbb{P}}^2$, since $\widetilde{L} \xrightarrow{\pi} \mathbb{P}^1$ is isomorphic. ($\widetilde{\mathbb{P}}^1_{p_1,p_2,p_3} = \mathbb{P}^1$)

$\widetilde{L}\cdot\widetilde{L} = (\pi^*L - E_1 - E_2 - E_3)\cdot(\pi^*L - E_1 - E_2 - E_3) = \pi^*L\cdot\pi^*L + E_1\cdot E_1 + E_2\cdot E_2 + E_3\cdot E_3$
$= 1 - 3 = -2$.

$\Rightarrow$ By the formula on P471, $\quad 0 = \pi(\widetilde{L}) = \dfrac{\widetilde{L}\cdot\widetilde{L} + K_S\cdot\widetilde{L}}{2} + 1$

$\Rightarrow$ $K_S\cdot\widetilde{L} = -2 + 2 = 0$. ⏎

$\widetilde{\mathbb{P}}^1 = \mathbb{P}^1$, for $\pi^{-1}(p_1) = \mathbb{P}^{1-1} = *$. $\Rightarrow$ $\widetilde{L} \to \mathbb{P}^1$ is one to one.