F is a diffeomorphism on $V \times \mathbb{C}^n$, for some nbd of x_0. since dF has non-zero determinant on V and F is one to one & onto. $V \subset U$.

$$\Rightarrow \cancel{F|_{f_V}} \quad F|_{\ker f_V} : \ker f_V \xrightarrow{\cong} V \times \mathbb{C}^{n-k}.$$

$\Rightarrow$ We have an open set V s.t

$$F|_{\ker f_V} : \ker f_V \longrightarrow V \times \mathbb{C}^{n-k} \subset V \times \mathbb{C}^n.$$

and $F : U \times \mathbb{C}^n \longrightarrow V \times \mathbb{C}^n$ is diffeomorphic.

In a similar way, $n \geq m$. by changing a base for $\mathbb{C}^m$, for a fixed $x_0 \in U$,

$$U \times \mathbb{C}^m \ni \left(x_0, \begin{pmatrix} 1 & 0 & & 0 & \cdots \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & 0 & \ddots \ 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} \right)$$

Consider a map G from $U \times \mathbb{C}^m$ to $U \times \mathbb{C}^m$ given by

$$U \times \mathbb{C}^m \xrightarrow{\quad G \quad} U \times \mathbb{C}^m$$

$$\left(x, \begin{pmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & & \\ a_{m1}(x) & \cdots & a_{mn}(x) \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} + \mathrm{II} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ z_{k+1} \\ \vdots \\ z_n \end{pmatrix} \right) \longmapsto \left(x, \begin{pmatrix} a_{11}(x) & \cdots & a_{1k}(x) & 0 & \cdots & 0 \\ \vdots & & & & & \\ a_{k1}(x) & \cdots & a_{kk}(x) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & & \\ \vdots & & & & \ddots & \\ 0 & \cdots & 0 & & & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_k \\ \vdots \\ z_n \end{pmatrix} \right)$$

$$\mathrm{II} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix}$$

At x_0, G is diffeomorphic. i.e $dG(x_0) = \text{identity}$. and furthermore for some small nbd of x_0, dG has no zero determinant. and G is one to one and onto.

$\Rightarrow$ G is diffeomorphic on $U \times \mathbb{C}^m$.

$\Rightarrow$ $\operatorname{Im} f_V \cong U \times \mathbb{C}^k$. and $G^{-1}|_{U \times \mathbb{C}^\ell} = \operatorname{Im} f_V$.