$\Rightarrow$ By Bott residue.

$$\int_{\mathbb{P}^n} c_1(\mathbb{P}^n)^{n-r} \cdot c_r(\mathbb{P}^n) = \sum_{i=0}^{n} \frac{P_1(A_{p_i})^{n-r} \cdot P_r(A_{p_i})}{\det(A_{p_i})}$$

$$P_1(A_{p_i})^{n-r} = \Big(\sum_{j \neq i} (\alpha_j - \alpha_i) \Big)^{n-r}$$

$$P_r(A_{p_i}) = P_r \begin{pmatrix} \alpha_0 - \alpha_i & 0 & \cdots & 0 & & & \\ 0 & \alpha_1 - \alpha_i & \cdots & 0 & & & \\ \vdots & \vdots & \ddots & & & & \\ 0 & 0 & \cdots & \alpha_{i-1} - \alpha_i & 0 & & \\ \vdots & \vdots & & 0 & \alpha_{i+1} - \alpha_i & & \\ & & & & & \ddots & \\ 0 & 0 & \cdots & & & & \alpha_n - \alpha_i \end{pmatrix}$$

$$= \sum_{\substack{\# I = r \\ i \notin I}} \Big(\prod_{j \in I} (\alpha_j - \alpha_i) \Big)$$

$$\Rightarrow \int_{\mathbb{P}^n} c_1(\mathbb{P}^n)^{n-r} \cdot c_r(\mathbb{P}^n) = \sum_{i=0}^{n} \frac{\Big(\sum_{j \neq i} (\alpha_j - \alpha_i) \Big)^{n-r} \Big(\sum_{\substack{\# I = r \\ i \notin I}} \prod_{j \in I} (\alpha_j - \alpha_i) \Big)}{\prod_{j \neq i} (\alpha_j - \alpha_i)}$$

$$= \sum_{i=0}^{n} \frac{(-(n+1)\alpha_i)^{n-r} \Big(\sum_{\# I = r,\, i \notin I} \prod_{j \in I} (\alpha_j - \alpha_i) \Big)}{\prod_{j \neq i} (\alpha_j - \alpha_i)}$$

$$= \sum_{i=0}^{n} \frac{(-1)^{n-r} (n+1)^{n-r} \alpha_i^{n-r} \cdot \sum_{\# I = r,\, i \notin I} \prod_{j \in I} (\alpha_j - \alpha_i)}{\prod_{j \neq i} (\alpha_j - \alpha_i)}$$

$\lrcorner\!\lrcorner$

Again, for $f(z)$ as above, $g(z) = z^{n-r} \sum_{\# I = r} \prod_{k \in I} (\alpha_k - z)$,

and $\varphi = (g/f)\, dz$.