Pf $d\eta = \bar\partial\eta + \partial\eta \Rightarrow d\eta = \partial\eta$ since η is holomorphic.
$\Rightarrow$ Again $\partial\eta$ is holomorphic & $d\partial\eta = dd\eta = 0$ (which is exact) $\Rightarrow$ By the previous argument, $\partial\eta = 0 = d\eta$. //

1. To show $b_{2q}(M) > 0$, we exhibit ω^q as a closed $2q$-form that is not exact: if $d\psi = \omega^q$,

then we have $\int_M \omega^n = \int_M d(\psi \wedge \omega^{n-q}) = 0$ (Remember always $d\omega = 0$)

But $\omega^n/n!$ is the volume form on M, and so this cannot happen.

3. The proof of 3 is clear: for V of complex dimension d, by the Wirtinger theorem from Section 2 above

$$\mathrm{vol}(V) = \frac{1}{d!}\int_V \omega^d \neq 0; \qquad \text{so} \quad (\eta_V) \neq 0 \text{ in } H_{2d}(V)$$

See P31. Q.E.D.

Note that 1 and 3 are extensions of the propositions proved on P64 for submanifolds of projective space.

The Hodge Identities and the Hodge Decomposition.

Let M be a compact complex manifold with hermitian metric ds^2, and associated $(1,1)$-form ω. We have defined a number of operators on the space $A^*(M)$ of differential forms on M, such as $\bar\partial, \partial, d, d^c$, their respective adjoints and associated Laplacians, and the decompositions

$$\Pi^{p,q} : A^* \longrightarrow A^{p,q}(M)$$
$$\Pi^r = \bigoplus_{p+q=r} \Pi^{p,q} : A^*(M) \longrightarrow A^r(M)$$

by type and degree. We define an additional operator $L : A^{p,q}(M) \longrightarrow A^{p+1,q+1}(M)$ by