We take
$$\bar\partial^*: A^{p,q}(E) \longrightarrow A^{p,q-1}(E)$$
to be given by
$$\bar\partial^* = -*_E\,\bar\partial\,*_E\,;$$
as before, $\bar\partial^*$ is the adjoint of $\bar\partial$, i.e for all $\varphi \in A^{p,q-1}(E)$ and $\psi \in A^{p,q}(E)$,
$$(\bar\partial\varphi, \psi) = (\varphi, \bar\partial^*\psi).$$

$\sqcap\!\sqcap$
$$(\bar\partial\varphi, \psi) = \int_M (\bar\partial\varphi \wedge *_E\psi) = (-1)^{p+q}\int_M \varphi\wedge\bar\partial *_E\psi$$

since $\bar\partial(\varphi\wedge *_E\psi) = d(\varphi\wedge *_E\psi) = \bar\partial\varphi\wedge *_E\psi + (-1)^{p+q-1}\varphi\wedge\bar\partial *_E\psi$, & $\int_M d(\varphi\wedge *_E\psi) = 0$ by stoke's theorem.

By $*_E^2 = (-1)^{p+q}$,
$$\int_M \varphi\wedge\bar\partial *_E\psi = \int_M \varphi\wedge(-1)^{p+q-1} *_E^2\,\bar\partial *_E\psi$$
$$= (-1)^{p+q-1}\int_M \varphi\wedge *_E(*_E\bar\partial *_E\psi) = (-1)^{p+q}\int_M \varphi\wedge *_E(-*_E\bar\partial *_E\psi)$$

since $\bar\partial *_E\psi \in A^{n-p,n-q+1}(E)$, $\Rightarrow *_E^2 = (-1)^{2n-p-q+1}$.

Thus
$$(\bar\partial\varphi, \psi) = (-1)^{p+q}\int_M \varphi\wedge\bar\partial *_E\psi = (-1)^{2(p+q)}\int_M \varphi\wedge *_E(-*_E\bar\partial *_E\psi)$$
$$= (\varphi, \bar\partial^*\psi) = \int_M \varphi\wedge *_E(\bar\partial^*\psi). \qquad \| \quad \langle\varphi, -*_E\bar\partial *_E\psi\rangle$$
$$\Rightarrow \quad \bar\partial^*\psi = -*_E\bar\partial *_E\psi$$

$\lrcorner\!\lrcorner$