Return to our compact, complex manifold M.

Hodge Theorem

1. $\dim \mathcal{H}^{p,q}(M) < \infty$ and
2. because of this, the orthogonal projection
$$\mathcal{H} : A^{p,q}(M) \longrightarrow \mathcal{H}^{p,q}(M)$$ is well-defined,

and $\exists$ a unique operator, the Green's operator,
$$G : A^{p,q}(M) \longrightarrow A^{p,q}(M).$$
with $G(\mathcal{H}^{p,q}(M)) = 0$, $\bar\partial G = G\bar\partial$, $\bar\partial^* G = G\bar\partial^*$ and

$$(**) . \qquad I = \mathcal{H} + \Delta G \quad \text{on } A^{p,q}(M)$$

The equation $(**)$ in the form
$$\psi = \mathcal{H}(\psi) + \bar\partial(\bar\partial^* G\psi) + \bar\partial^*(\bar\partial G\psi)$$ is called
the Hodge decomposition on forms, since it directly implies
the orthogonal direct-sum decomposition
$$A^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \bar\partial A^{p,q-1}(M) \oplus \bar\partial^* A^{p,q+1}(M).$$

Γ $\langle \bar\partial\psi, \bar\partial^*\phi \rangle = 0 = \langle \bar\partial\bar\partial\psi, \phi \rangle \Rightarrow \bar\partial A^{p,q-1} \cap \bar\partial^* A^{p,q+1} = 0.$

Suppose $\psi \in \mathcal{H}^{p,q}(M) \cap \bar\partial A^{p,q-1}(M)$.

$\Rightarrow \psi = \bar\partial\phi . \quad \phi \in A^{p,q-1} \quad \Rightarrow$ Since $\Delta\psi = 0$, $\bar\partial^*\psi = \bar\partial\psi = 0$,

$\langle \bar\partial\phi, \bar\partial\phi \rangle = \langle \psi, \psi \rangle = \langle \psi, \bar\partial\phi \rangle = \langle \bar\partial^*\psi, \phi \rangle = 0 .$

$\Rightarrow \bar\partial\phi = \psi = 0 \quad \Rightarrow \quad \mathcal{H}^{p,q}(M) \cap \bar\partial A^{p-q}(M) = 0 .$

$\psi \in \mathcal{H}^{p,q}(M) \cap \bar\partial^* A^{p,q+1} \quad \Rightarrow \quad \psi = \bar\partial^*\phi \quad \bar\partial\psi = 0$

$\langle \bar\partial^*\phi, \bar\partial^*\phi \rangle = \langle \psi, \psi \rangle = \langle \psi, \bar\partial^*\phi \rangle = \langle \bar\partial\psi, \phi \rangle = 0$

$\Rightarrow \psi = \bar\partial^*\phi = 0 \quad \Rightarrow \mathcal{H}^{p,q} \cap \bar\partial^* A = 0$