Suppose $\exists\ x' \neq x$ s.t $\tilde{\pi}(x) = \tilde{\pi}(x') = q$. Consider $\overline{pq}$.
$\Rightarrow \#(\overline{pq} \cap S) \geq 3 \Rightarrow \overline{pq} \subset S \Rightarrow \overline{pq} \subset T_p'(S) \cap S = L_1 \cup L_2$
$\Rightarrow \overline{pq} = L_1$ or $L_2 \Rightarrow q = q_1$ or $q_2 \Rightarrow$ Contradiction.
$\Rightarrow \tilde{\pi} : \tilde{S} - \tilde{L}_1 - \tilde{L}_2 \longrightarrow H - \{q_1, q_2\}$ is one to one and onto.
$\Rightarrow \tilde{\pi} : \tilde{S} \longrightarrow H$ is the blow-up of H at q_1 and q_2. ⌟

Thus we may obtain $\mathbb{P}^2$ from a quadric surface S by blowing up one point p of S and blowing down the proper transforms of the two lines on S through p.

⌜ Blowing down means "some sort of" collapsing. ⌟

In reverse: we may obtain a quadric surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ from $\mathbb{P}^2$ by blowing up two points q_1, q_2 on $\mathbb{P}^2$ and blowing down the proper transform of the line $\overline{q_1 q_2} \subset \mathbb{P}^2$.

⌜ $\tilde{S} \xrightarrow{\tilde{\pi}} \mathbb{P}^2$ is the blow-up at q_1 and q_2.
$\tilde{S} \longrightarrow S$ is the blow-up at p.
$\Rightarrow \tilde{\pi}(E) = \overline{q_1 q_2} \Rightarrow \tilde{\pi}^{-1}(\overline{q_1 q_2}) = E \cup \tilde{L}_1 \cup \tilde{L}_2$
$\Rightarrow$ The proper transform of $\overline{q_1 q_2} = E$. $\Rightarrow$ By blowing down E, we obtain S. ⌟

We will see this operation more explicitly following our discussion of the cubic surface.

Note that since the only invariant of a symmetric