all six points P_i, then the curve $\pi^{-1}(C) - E_1 - E_2 - \cdots - E_6$ is certainly in the linear system $|\tilde{C}|$; conversely, if D is any curve in $|\tilde{C}|$, then

$$\pi(D)\cdot H = D\cdot\pi^* H = 3$$

so $\pi(D)$ is a cubic curve, and

$$D\cdot E_i = -E_i\cdot E_i = 1$$

so D meets every exceptional divisor E_i, i.e., $\pi(D)$ passes through all six points P_i.

$\ulcorner$ C is cubic. $\Rightarrow$ $\deg C = 3$ $\Rightarrow$ $[C] = [3H]$, since $C\cdot H = 3 = 3H\cdot H$.

$\Rightarrow$ $\pi^{-1}(C) - E_1 - \cdots - E_6 \sim \pi^*(3H) - E_1 - \cdots - E_6$

p476 $\pi(D)\cdot H = (3H)\cdot H = 3$ since $D \sim \tilde{C} = \pi^* 3H - E_1 - \cdots - E_6$

$D\cdot\pi^* H = \tilde{C}\cdot\pi^* H = (\pi^* 3H - E_1 - \cdots - E_6) = \pi^* H = \pi^* 3H \cdot \pi^* H$
$= 3H\cdot H = 3$. $\Rightarrow$ $\pi(D)$ is a cubic curve.

$D\cdot E_i = \tilde{C}\cdot E_i = (\pi^* 3H - E_1 - \cdots - E_6)\cdot E_i = -E_i\cdot E_i = 1$ $\lrcorner$

Thus the system $|\tilde{C}|$ consists exactly of curves
$\pi^{-1}(C) - E_1 - E_2 \cdots - E_6$,
where C is a cubic plane curve containing $P_1, P_2, \cdots P_6$.

$\ulcorner$ $D - \pi^*(\pi(D)) = a_1 E_1 + \cdots + a_6 E_6$ for a curve $D \in |\tilde{C}|$

$\Rightarrow$ $(D - \pi^*(\pi(D)))\cdot E_i = D\cdot E_i - \pi^*(\pi(D))\cdot E_i = 1 - \pi(D)\cdot P_i$
$= 1 - 0 = 1 = -a_i$ $\Rightarrow$ $a_i = -1$

$\Rightarrow$ $D = \pi^*(\pi(D)) - E_1 - E_2 - \cdots - E_6$, $\pi(D) = C$ is a