$$\xi = \xi_{\alpha 1} e_1 + \xi_{\alpha 2} e_2 + \cdots + \xi_{\alpha n} e_n$$

$$\begin{aligned} D\xi = D\xi_{\alpha i} e_i &= d\xi_{\alpha i} \otimes e_i + \xi_{\alpha i} \cdot D e_i \\ &= d\xi_{\alpha i} \otimes e_i + \xi_{\alpha i} \, \theta_{\alpha ij} \otimes e_{\alpha j} \\ &= d\xi_{\alpha i} \otimes e_i + \xi_{\alpha j} \, \theta_{\alpha ji} \otimes e_{\alpha i} \\ &= (d\xi_{\alpha i} + \xi_{\alpha j} \, \theta_{\alpha ji}) \otimes e_{\alpha i} \\ &= (d\xi_{\alpha i} + {}^t(\theta_{\alpha ij}) \xi_{\alpha j}) \otimes e_{\alpha i} \end{aligned}$$

If $\varphi_\beta : E|U_\beta \to U_\beta \times \mathbb{C}^n$ is another trivialization of E over $U_\beta \subset M$ with $\varphi_\alpha = g_{\alpha\beta} \varphi_\beta$ and θ_β is the connection matrix for D in terms of φ_β, then

$$\theta_\alpha = g_{\alpha\beta} \cdot \theta_\beta \cdot g_{\alpha\beta}^{-1} + dg_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}.$$

See P72 ⊛

Note that the dependence of θ on the choice of frame is nonlinear — i.e., θ is not a tensor field of E. Indeed, by solving the equations $g_{\alpha\beta}(x_0) = $ identity and $dg_{\alpha\beta}(x_0) = -\theta_\alpha(x_0)$, we can find a trivialization of E in a nbd of any point $x_0 \in M$ in terms of which the connection matrix $\theta_\beta(x_0)$ of D vanishes at x_0.

$U_\alpha = U_\beta$

Given a trivialization φ_α, if we find $g_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n)$, s.t $g_{\alpha\beta}(x_0) = id$ & $dg_{\alpha\beta}(x_0) = -\theta_\alpha(x_0)$, where $\theta_\alpha(x_0)$ is given by the φ_α,
then, by considering the trivialization $\varphi_\beta = g_{\alpha\beta} \cdot \varphi_\alpha$, and the relation

$$\theta_\alpha = g_{\alpha\beta} \cdot \theta_\beta \cdot g_{\alpha\beta}^{-1} + dg_{\alpha\beta} \cdot g_{\alpha\beta}^{-1},$$

$$\theta_\alpha(x_0) = \theta_\beta(x_0) + dg_{\alpha\beta}(x_0) = \theta_\beta(x_0) - \theta_\alpha(x_0) = 0.$$