$$= X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot (x_i - \alpha_i) = X_0 \cdot \sum_{i=1}^{n} \frac{\partial f_\alpha}{\partial x_i}(p) \cdot (x_i - \alpha_i),$$

so the homogeneous form describes the same subspace.

Pf Since $f_\alpha(x_1, \cdots x_n) = \frac{1}{X_0^d} F_\alpha(X_0, \cdots X_n)$, $\quad i \neq 0$

$$\frac{\partial}{\partial X_i}\left(\frac{1}{X_0^d} F_\alpha(X_0, X_1, \cdots X_n)\right) = \frac{1}{X_0^d} \frac{\partial F_\alpha}{\partial X_i} = \frac{\partial}{\partial X_i} f_\alpha\left(\frac{X_1}{X_0}, \frac{X_2}{X_0}, \cdots \frac{X_n}{X_0}\right)$$

$$= \lim_{h \to 0} \frac{f_\alpha\left(\frac{X_1}{X_0}, \cdots \frac{X_i + h}{X_0}, \cdots \frac{X_n}{X_0}\right) - f_\alpha\left(\frac{X_1}{X_0}, \cdots \frac{X_i}{X_0}, \cdots \frac{X_n}{X_0}\right)}{h}$$

$$= \lim_{h \to 0} \frac{f_\alpha\left(\frac{X_1}{X_0}, \cdots \frac{X_i + h}{X_0}, \cdots \frac{X_n}{X_0}\right) - f_\alpha\left(\frac{X_1}{X_0}, \cdots \frac{X_i}{X_0}, \cdots \frac{X_n}{X_0}\right)}{\frac{h}{X_0}} \cdot \frac{1}{X_0}$$

$$= \frac{1}{X_0} \lim_{h \to 0} \frac{f_\alpha(x_1, \cdots x_i + \frac{h}{X_0} \cdots x_n) - f_\alpha(x_1, \cdots x_n)}{h / X_0} = \frac{1}{X_0} \frac{\partial f_\alpha}{\partial x_i}$$

$$\Rightarrow \frac{1}{X_0^d} \frac{\partial F_\alpha}{\partial X_i} = \frac{1}{X_0} \frac{\partial f_\alpha}{\partial x_i}. \quad \Rightarrow \frac{1}{X_0^{d-1}} \frac{\partial F_\alpha}{\partial X_i} = \frac{\partial f_\alpha}{\partial x_i}, \quad i = 1, 2, \cdots n.$$

$p \in V \Rightarrow p = [1, \alpha_1, \alpha_2 \cdots \alpha_n] \in V \subset \mathbb{P}^n.$

$F_\alpha(t, \alpha_1 t, \cdots \alpha_n t) = 0$

$$\frac{d}{dt} F_\alpha(t, \alpha_1 t, \cdots \alpha_n t) = \frac{\partial F_\alpha}{\partial X_0}(t, \cdots \alpha_n t)\, 1 + \frac{\partial F_\alpha}{\partial X_1} \alpha_1 + \cdots \frac{\partial F_\alpha}{\partial X_n}(t, \cdots \alpha_n t) \alpha_n$$

$= 0 \Rightarrow$ Plug $t = 1$, $\frac{\partial F_\alpha}{\partial X_0}(p) + \frac{\partial F_\alpha}{\partial X_1} \alpha_1 + \cdots + \frac{\partial F_\alpha}{\partial X_n} \alpha_n = \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i} \alpha_i$

$= 0$, where, $\alpha_i = 1$, $i = 0$.

$$\Rightarrow X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot x_i = X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p) \cdot x_i - X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i} \alpha_i$$