The limiting position of $Z[r]$ as $r \downarrow 0$ is the tangent cone $C(Z)$ to Z at the origin, and by the Wirtinger theorem applied this time to the projective space $\mathbb{P}^{n-1}$,

$$\int_{C(Z)} \Omega^{n-1} = \text{degree}\,(C(Z))$$
$$= \text{mult}_{\{0\}}(Z).$$

Q.E.D.

$$\left| \int_{B_{n-1}[r]} \frac{\bar{z}_j\, dz_j \wedge z_i\, d\bar{z}_i}{\|Z\|^4} \right| \leq c \int_{B_{n-1}[r]} \frac{\rho^2}{\rho^4} = c \int_{B_{n-1}[r]} \frac{1}{\rho^2}$$

$$= C \int_0^r \int_{\partial B_{n-1}[\rho]} \frac{1}{\rho^2}\, dV dr$$

$\longrightarrow 0$ as $r \to 0$ if $n-1 \geq 2$.

$$\int_{C(Z)} \Omega^{n-1} = \text{"Vol}\,(C(Z))\text{"} = \#(Z \cdot \mathbb{C}) = \#\text{ of some points.}$$

What does degree $(C(Z))$ mean?

$\int_{C(Z)} \Omega^{n-1}$ does not make sense, but I can believe

$$\lim_{r \to 0} \int_{Z[r]} \Omega^{n-1} = \text{mult}_{\{0\}}(Z).$$

The proof is not too difficult to make precise, and it is essentially obvious in case the origin is a smooth point of Z, which is all we shall use.

In case the origin is a smooth point of Z, then