

$$\mathbb{R} \quad \sigma = C_n \frac{*rdr}{r^n} \Rightarrow d\sigma = C_n d\left(\frac{*rdr}{r^n}\right) = C_n d\left(\frac{1}{r^n}\right) \wedge (*rdr) \\ = C_n (-n) r^{-n-1} dr \wedge (*rdr) = -\frac{nC_n}{r^{n+1}} dr \wedge (*rdr).$$

We had better use  $\sigma = C_n \frac{\sum \Phi_i(x)}{r^n}$ .

$$\begin{aligned} d\sigma &= C_n d\left(\frac{\sum \Phi_i}{r^n}\right) = C_n dr^{-n} \wedge \sum \Phi_i + \frac{1}{r^n} d\sum \Phi_i \cdot C_n \\ &= C_n (-n) r^{-n-1} \sum \frac{x_i}{r} dx_i \wedge \sum \Phi_i + \frac{1}{r^n} \sum d\Phi_i \cdot C_n \\ &= C_n (-n) \left(\sum \frac{x_i dx_i}{r^{n+2}}\right) \wedge (*rdr) + \frac{1}{r^n} \sum \Phi_i(x) \cdot C_n \\ &= -C_n \frac{n}{r^{n+2}} r dr \wedge (*rdr) + \frac{n \Phi(x)}{r^n} \cdot C_n \\ &= \left(\frac{n \Phi(x)}{r^n} - \frac{nrdr \wedge (*rdr)}{r^{n+2}}\right) C_n = C_n \left(\frac{n \Phi(x)}{r^n} - \frac{n \sum x_i dx_i}{r^{n+2}}\right) \\ &\quad \wedge \sum (-1)^{i-1} x_i dx_1 \wedge \dots \wedge \hat{dx}_i \wedge \dots \wedge dx_n \Big) = C_n \left(\frac{n \Phi(x)}{r^n} - \frac{n r^2 \Phi}{r^{n+2}}\right) = 0. \end{aligned}$$

By Stokes' theorem, then, the integral

$$\int_{\|x\|=\epsilon} \sigma$$

of  $\sigma$  over a sphere is independent of the radius  $\epsilon > 0$ .

$$\mathbb{R} \quad 0 = \int_{B(0,\epsilon_1) - B(0,\epsilon_2)} d\sigma = \int_{\partial B(0,\epsilon_1) - \partial B(0,\epsilon_2)} \sigma \quad \text{by Stokes' theorem.} \quad \Downarrow$$

Consequently, choosing  $C_n$  properly,  $\sigma$  is the unique form on  $\mathbb{R}^n - \{0\}$  that is invariant under proper rotations,