Let $C_n = (-1)^{n(n-1)/2}\left(\frac{i}{\cancel{2}}\right)^n$. and

$\Phi = \frac{\omega^n}{n!} = C_n \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_n$. be the volume form on M associated to the metric. The global inner product

$$(\psi, \eta) = \int_M (\psi(z), \eta(z))\, \Phi(z)$$ makes the space $A^{p,q}(M)$

into a pre-Hilber space. We pose the question: Given a $\bar\partial$-closed form $\psi \in Z_{\bar\partial}^{p,q}(M)$, among all the forms $\{\psi + \bar\partial\eta\}$ representing the Dolbeault cohomology class $[\psi] \in H_{\bar\partial}^{p,q}(M)$ of φ, can we find one of smallest norm? To do this, assume for a moment that $A^{p,q}(M)$ is complete and $\bar\partial$ is bounded, and define the adjoint operator

$$\bar\partial^* : A^{p,q}(M) \longrightarrow A^{p,q-1}(M)$$ by requiring

$$(\bar\partial^*\psi, \eta) = (\psi, \bar\partial\eta). \quad \text{for all } \eta \in A^{p,q-1}(M).$$

This will be justified in a moment, but first we show

Lemma. A $\bar\partial$-closed form $\psi \in Z_{\bar\partial}^{p,q}(M)$ is of minimal norm in $\psi + \bar\partial A^{p,q-1}(M) \Leftrightarrow \bar\partial^*\psi = 0$.

pf). $\bar\partial^*\psi = 0 \Rightarrow$ for any $\eta \in A^{p,q-1}(M)$ with $\bar\partial\eta \neq 0$

$$\|\psi + \bar\partial\eta\|^2 = (\psi + \bar\partial\eta, \psi + \bar\partial\eta) = \|\psi\|^2 + \|\bar\partial\eta\|^2$$
$$+ 2\,\mathrm{Re}(\psi, \bar\partial\eta) = \|\psi\|^2 + \|\bar\partial\eta\|^2 + 2\,\mathrm{Re}(\overset{=0}{\cancel{\bar\partial^*\psi}}, \eta)$$
$$= \|\psi\|^2 + \|\bar\partial\eta\|^2 > \|\psi\|^2,$$ so ψ has minimal norm.

Conversely, if ψ has minimal norm, then for any $\eta \in A^{p,q-1}(M)$,

$$\frac{\partial}{\partial t}\|\psi + t\bar\partial\eta\|^2(0) = 0.$$