Next we define the coboundary map

$$\delta^*: H^p(M, \mathcal{G}) \longrightarrow H^{p+1}(M, \mathcal{E}) \quad \text{as follows:}$$

Given $\sigma \in C^p(\underline{\mathcal{U}}, \mathcal{G})$ with $\delta\sigma = 0$,
(prove later) we can always pass to a refinement $\underline{\mathcal{U}}'$ of $\underline{\mathcal{U}}$ and find $\tau \in C^p(\underline{\mathcal{U}}', \mathcal{F})$ s.t $\beta(\tau) = \rho\sigma$, i.e.

$$\begin{tikzcd}
0 \arrow[r] & C^p(\underline{\mathcal{U}}, \mathcal{E}) \arrow[r, "\alpha"] & C^p(\underline{\mathcal{U}}, \mathcal{F}) \arrow[r, "\beta"] & C^p(\underline{\mathcal{U}}, \mathcal{G}) \arrow[r] \arrow[dd, "\rho"'] & 0 \\
& & & \ni \sigma & \\
0 \arrow[r] & C^p(\underline{\mathcal{U}}', \mathcal{E}) \arrow[r, "\alpha"] & C^p(\underline{\mathcal{U}}', \mathcal{F}) \arrow[r, "\beta"] & C^p(\underline{\mathcal{U}}', \mathcal{G}) \arrow[r] & 0 \\
& & \exists\ \underline{\underline{\tau}} \arrow[r, mapsto] & \rho\sigma &
\end{tikzcd}$$

Then $\beta\delta\tau = \delta\beta\tau = \delta\rho\sigma = \rho\delta\sigma = 0$, so by passing to a further refinement $\underline{\mathcal{U}}''$, we can find $\mu \in C^{p+1}(\underline{\mathcal{U}}'', \mathcal{E})$ s.t

$\alpha\mu = \rho\delta\tau$.

$$\begin{tikzcd}
0 \arrow[r] & C^p(\underline{\mathcal{U}}', \mathcal{E}) \arrow[r, "\alpha"] & C^p(\underline{\mathcal{U}}', \mathcal{F}) \arrow[r, "\beta"] & C^p(\underline{\mathcal{U}}', \mathcal{G}) \arrow[r] & 0 \\
& & \tau \arrow[r, mapsto] \arrow[d, "\delta"'] & \rho\sigma \arrow[d, "\delta"] & \\
0 \arrow[r] & C^{p+1}(\underline{\mathcal{U}}', \mathcal{E}) \arrow[r, "\alpha"] \arrow[dd] & C^{p+1}(\underline{\mathcal{U}}', \mathcal{F}) \arrow[r, "\beta"] & C^{p+1}(\underline{\mathcal{U}}', \mathcal{G}) \arrow[r] & 0 \\
& & \delta\tau \arrow[r, mapsto] \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & C^{p+1}(\underline{\mathcal{U}}'', \mathcal{E}) \arrow[r, "\alpha"] & C^{p+1}(\underline{\mathcal{U}}'', \mathcal{F}) \arrow[r, "\beta"] & C^{p+1}(\underline{\mathcal{U}}'', \mathcal{G}) \arrow[r] & 0 \\
& \mu \arrow[r, mapsto] & \rho\delta\tau \arrow[r, mapsto] & 0 &
\end{tikzcd}$$