cubic curve containing $p_1 \cdots p_6$. ⌟

We claim now that the linear system $|\tilde{C}|$ embeds $\tilde{\mathbb{P}}^2$ as a cubic surface in $\mathbb{P}^3$. This involves quite a bit of checking: we have to show that

1. $\tilde{C} \cdot \tilde{C} = 3$
2. $\dim |\tilde{C}| = 3$
3. $\iota_{\tilde{C}}$ seperates points $p \neq q \in \tilde{\mathbb{P}}^2$, for
   a. $p, q \in \tilde{\mathbb{P}}^2 - \cup E_i$,
   b. $p \in E_i$, $q \in \tilde{\mathbb{P}}^2 - \cup E_i$
   c. $p \in E_i$, $q \in E_j$, and
   d. $p, q \in E_i$; and
4. $\iota_{\tilde{C}}$ has nonzero differential at $p$ for
   a. $p \in \tilde{\mathbb{P}}^2 - \cup E_i$, and
   b. $p \in E_i$.

Assertion 1 is immediate: we have

$$\tilde{C} \cdot \tilde{C} = \pi^* 3H \cdot \pi^* 3H + E_1 \cdot E_1 + \cdots + E_6 \cdot E_6 = 9 - 6 = 3.$$

The other assertions, however, are of a different character. For example, we know that the complete linear system of cubic curves in the plane has dimension 9, and that the requirement that a cubic pass through any one of the points $p_i$ imposes one linear condition on the system $|3H|$; the statement $\dim |\tilde{C}| = 3$ amounts to saying that the conditions imposed by the six points $p_i$ are independent.