$\mathrm{Ext}^1_{\mathfrak{o}}(M, E) = 0$ for all projective modules $E \Leftrightarrow M$ is projective.

Proof. Suppose that $\mathrm{Ext}^1_{\mathfrak{o}}(M, E)$ for all projective (= free) modules E. Choosing generators for M, we obtain a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] & E \arrow[r] \arrow[l, dashed, bend right, "\pi"'] & M \arrow[r] & 0 \end{tikzcd}$$

with E free. Applying the other long exact sequence of Ext's gives

$$\mathrm{Hom}_{\mathfrak{o}}(E, E) \longrightarrow \mathrm{Hom}_{\mathfrak{o}}(R, E) \longrightarrow \mathrm{Ext}^1_{\mathfrak{o}}(M, E) = 0.$$

Consequently the dotted arrow π exists, and $M \cong \ker \pi$, $E \cong R \oplus M$.

「 I think that "Consequently the dotted arrow π exists" is not correct. I don't know how to fix it.」

Since a direct summand of a projective module is again projective, we are done. Q.E.D.

「 $P = A \oplus B$, P projective.

$$\begin{tikzcd} A \arrow[d, dashed, "?\ \beta"'] \arrow[dr, "\alpha"] & & \\ N \arrow[r, "f"] & Q \arrow[r] & 0 \end{tikzcd}$$

Consider $\alpha' : P \longrightarrow Q$ defined by $\alpha'(a, b) = \alpha(a)$.

$\Rightarrow$ By the definition of projective, $\exists\ \beta' : P \longrightarrow N$ s.t. $f \circ \beta = \alpha'$. $\Rightarrow$ Let $\beta(a) = \beta'(a, 0)$.

$\Rightarrow$ $f \circ \beta(a) = f \circ \beta'(a, 0) = \alpha'(a, 0) = \alpha(a)$ $\Rightarrow$ A is projective. Q.E.D.」

In closing we should like to comment that the name