By the Hodge theorem,

$$0 = \mathcal{H}^{0,q}(M) \cong H_{\bar\partial}^{0,q}(M), \quad q>0 .$$

This is a special case of the famous Kodaira vanishing theorem, for which the general argument will be given in Section 3 of Chapter 1.

Applications of the Hodge Theorem

We begin by noting the isomorphism

$$\mathcal{H}^{p,q}(M) \longrightarrow H_{\bar\partial}^{p,q}(M)$$

between the harmonic space and Dolbeault cohomology groups. In fact, by the Hodge decomposition every $\bar\partial$ closed form $\psi \in Z_{\bar\partial}^{p,q}(M)$ is

$$\psi = \mathcal{H}(\psi) + \bar\partial(\bar\partial^* G\psi)$$

since $\bar\partial G\psi = G(\bar\partial\psi) = 0$. Combining this isomorphism with the Dolbeault isomorphism, we find

$$\mathcal{H}^{p,q}(M) \xrightarrow{\sim} H^q(M, \Omega^p).$$

By the first statement in the Hodge theorem, this implies

Finite Dimensionality

$$\dim H^q(M, \Omega^p) < \infty.$$

It is instructive to give a direct proof of finite dimensionality in the case $q=0$.
Let $\{U_i\}$ be a finite coordinate covering of M with holomorphic coordinates $z_{i,1}, \ldots z_{i,n}$ in U_i.

We may find relatively compact open subsets $V_i \subset U_i$ that still constitute a covering of M. A global section