simplicial decomposition of M and the complex $(\tilde{C}^*, \delta)$ of cochains in the dual cell decomposition.

$$C_k(M) = \bigoplus_{\alpha} \langle \sigma_\alpha^k \rangle$$ where $\langle \sigma_\alpha^k \rangle$ is the free abelian group generated by σ_α^k.

$$\begin{tikzcd} \Rightarrow C_k(M) \arrow[r] & H_{n-k}(M^{n-k}, M^{n-k-1}) \arrow[r, "\cong"] & \mathrm{Hom}(H_{n-k}(M^{n-k}, M^{n-k-1}), Z) \\ \sigma_\alpha^k \arrow[r, mapsto] & \Delta_\alpha^{n-k} \arrow[r, mapsto] & \tilde{\Delta}_\alpha^{n-k} \end{tikzcd}$$

where M^{n-k} is $(n-k)$-skeleton of dual cell decomposition.

Put $\tilde{C}^{n-k}(M) = \mathrm{Hom}(H_{n-k}(M^{n-k}, M^{n-k-1}); Z) \cong H_{n-k}(M^{n-k}, M^{n-k-1})$

$\cong \bigoplus_{\#\text{ of }(n-k)\text{-cells}} Z$ See Milnor & Stasheff. p263 Th A.4

$$\begin{array}{ccc} \mathrm{Ker}\, \partial & \longrightarrow & \mathrm{ker}\, \delta \\ \cup & & \\ \cancel{\sum \sigma_\alpha^k} & & \\ \sum a_\alpha \sigma_\alpha^k & \longmapsto & \sum a_\alpha * \sigma_\alpha^k \end{array}$$

$$\Rightarrow \quad \delta(\textstyle\sum a_\alpha * \sigma_\alpha^k) = \sum a_\alpha \delta(* \sigma_\alpha^k) = \sum a_\alpha \delta(\Delta_\alpha^{n-k})$$

$$= \textstyle\sum a_\alpha (-1)^{n-k+1} * \partial(\sigma_\alpha^k) = (-1)^{n-k+1} (\partial(\sum a_\alpha \sigma_\alpha^k)) = 0.$$

The resulting isomorphism

$$D: H_k(M, \mathbb{Z}) \longrightarrow H^{n-k}(M, \mathbb{Z})$$

have the property that $\#(\gamma \cdot \lambda) = D\gamma(\lambda)$

for any $\gamma \in H_k(M, \mathbb{Z})$ and $\lambda \in H_{n-k}(M, \mathbb{Z})$; and the theorem follows;