$$\left.\frac{d}{dt}\right|_{t=0} \exp t\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{pmatrix} = CAC^{-1}$

Jordan Canonical Form.

$$\det(e^A) = \det(C e^A C^{-1}) = \det\left(I + CAC^{-1} + \frac{(CAC^{-1})^2}{2!} + \cdots + \frac{(CAC^{-1})^n}{n!} + \cdots\right.$$

$$= \det\left(I + \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda_1^2 & * \\ 0 & \lambda_2^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda_1^3 & * \\ 0 & \lambda_2^3 \end{pmatrix} + \cdots + \frac{1}{n!}\begin{pmatrix} \lambda_1^n & * \\ 0 & \lambda_2^n \end{pmatrix} + \cdots\right.$$

$$= \det\begin{pmatrix} 1+\lambda_1+\frac{1}{2!}\lambda_1^2+\cdots\frac{1}{n!}\lambda_1^n+\cdots & * \\ 0 & 1+\lambda_2+\frac{1}{2!}\lambda_2^2+\cdots+\frac{1}{n!}\lambda_2^n+\cdots \end{pmatrix}$$

$$= \det\begin{pmatrix} e^{\lambda_1} & * \\ 0 & e^{\lambda_2} \end{pmatrix} = e^{\lambda_1}e^{\lambda_2} = e^{\lambda_1+\lambda_2} = e^{\operatorname{trace} A}$$

See Warner. P107 for $e^{\operatorname{trace} A} = \det(e^A)$.

We take as standard generators

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

with the relations

$$[X, Y] = H, \quad [H, X] = 2X \quad [H, Y] = -2Y.$$

Now, let V be a finite dimensional complex vector space, $gl(V)$ its algebra of endomorphisms. We want to study Lie Algebra maps

$$\rho : sl_2 \longrightarrow gl(V),$$

i.e, linear maps ρ such that

$$\rho([A, B]) = \rho(A)\rho(B) - \rho(B)\rho(A).$$

Such a map is called a representation of sl_2 in V; V is called an sl_2-module.