$\sqrt{-1}\ b^\nu_{\alpha k}\, a^\nu_{\beta k}\, dx_\alpha \wedge dx_\beta = \sqrt{-1} a^\nu_{\alpha k}\, b^\nu_{\beta k}\, dx_\beta \wedge dx_\alpha = -\sqrt{-1}\, a^\nu_{\alpha k}$
$b^\nu_{\beta k}\, dx_\alpha \wedge dx_\beta \quad \Rightarrow \quad df_k \wedge d\bar{f}_k (P_\nu) = -2\sqrt{-1}\, a^\nu_{\alpha k} b^\nu_{\beta k}\, dx_\alpha \wedge dx_\beta.$

$$\frac{\sqrt{-1}}{2} df_k \wedge d\bar{f}_k = \frac{\sqrt{-1}}{2}(-2)\sqrt{-1} \sum a^\nu_{\alpha k} b^\nu_{\beta k}\, dx_\alpha \wedge dx_\beta$$
$$= a^\nu_{\alpha k} b^\nu_{\beta k}\, dx_\alpha \wedge dx_\beta.$$

$\Rightarrow \Phi\, df_1 \wedge d\bar{f}_1 \wedge \cdots \wedge df_n \wedge d\bar{f}_n$

$= \Phi \sum a^\nu_{\alpha 1} b^\nu_{\beta 1}\, dx_\alpha \wedge dx_\beta \wedge \cdots \wedge \sum a^\nu_{\alpha n} b^\nu_{\beta n}\, dx_\alpha \wedge dx_\beta$

$= \Phi \sum a^\nu_{\alpha 1} a^\nu_{\beta n+1}\, dx_\alpha \wedge dx_\beta \wedge \cdots \wedge \sum a^\nu_{\alpha n} a^\nu_{\beta 2n}\, dx_\alpha \wedge dx_\beta$

(Here we put $b^\nu_{\beta k} = a^\nu_{\beta, n+k}$)

$= \Phi \sum_\sigma a^\nu_{\sigma(1) 1}\, a^\nu_{\sigma(n+1), n+1}\, dx_{\sigma(1)} \wedge dx_{\sigma(n+1)} \wedge \cdots\cdot a^\nu_{\sigma(n) n}\, a^\nu_{\sigma(2n), 2n}\, dx_{\sigma(n)} \wedge dx_{\sigma(2n)}$

$= \Phi \sum_\sigma a^\nu_{\sigma(1) 1}\, a^\nu_{\sigma(2), 2} \cdots a^\nu_{\sigma(2n), 2n}\ dx_{\sigma(1)} \wedge dx_{\sigma(n+1)} \wedge dx_{\sigma(2)} \wedge dx_{\sigma(n+2)} \wedge \cdots$
$\wedge\ dx_{\sigma(n)} \wedge dx_{\sigma(2n)}$

$= \Phi \sum_\sigma a^\nu_{\sigma(1), 1} \cdots a^\nu_{\sigma(2n), 2n}\ dx_{\sigma(1)} \wedge dx_{\sigma(2)} \wedge \cdots \wedge dx_{\sigma(2n)} (-1)^{1+2+\cdots+(n-1)}$

$= \Phi \sum_\sigma a^\nu_{\sigma(1), 1} \cdots a^\nu_{\sigma(2n), 2n}\ \epsilon(\sigma)\ dx_1 \wedge \cdots \wedge dx_{2n}\ (-1)^{\frac{n(n-1)}{2}}$

$= \Phi\ \det(A_{P_\nu})\ (-1)^{\frac{n(n-1)}{2}}\ dx_1 \wedge \cdots \wedge dx_{2n} = \mathrm{Sgn}\ dx_1 \wedge \cdots \wedge dx_{2n}$
$\Rightarrow \Phi$ has the sign as $\mathrm{sgn} \det(A_{P_\nu})\ (-1)^{\frac{n(n-1)}{2}}$.

Thus by Gauss-Bonnet II we have
$$C_n(E) = \sum (-1)^{n(n-1)/2}\ \mathrm{sgn} \det(A_{P_\nu}).$$

$\ulcorner$ $C_n(E)$ is Poincare dual to the degeneracy cycle
$D_{n-n+1} = D_1 = \sum_{P_\nu} (-1)^{\frac{n(n-1)}{2}} \mathrm{sgn} \det(A_{P_\nu}) \cdot [P_\nu]$.

$$\Rightarrow C_n(E) = \sum_{P_\nu} (-1)^{\frac{n(n-1)}{2}}\ \mathrm{sgn} \det(A_{P_\nu}).$$