s.t $a_1\tilde{\sigma}_1(x') + a_2\tilde{\sigma}_2(x') + a_3\tilde{\sigma}_3(x') + a_4\tilde{\tau}(x') = 0$.

$\Rightarrow \quad [(\tilde{\sigma}_1(x), \tilde{\sigma}_2(x), \tilde{\sigma}_3(x), \tilde{\tau}(x))]$
$= [(\tilde{\sigma}_1(x'), \tilde{\sigma}_2(x'), \tilde{\sigma}_3(x'), \tilde{\tau}(x'))]$

since $\langle(\tilde{\sigma}_1(x), \tilde{\sigma}_2(x), \tilde{\sigma}_3(x), \tilde{\tau}(x))\rangle^{\perp} = \langle(\tilde{\sigma}_1(x'), \tilde{\sigma}_2(x'), \tilde{\sigma}_3(x'), \tilde{\tau}(x'))\rangle^{\perp}$.

$\Rightarrow \quad \tilde{f}| : E_1 \longrightarrow \tilde{f}(E_1)$ is two to one.

$\Rightarrow \quad \#(\tilde{f}(E_1) \cap \mathbb{P}^2) = 1$, since $\tilde{f}|$ is two to one and $\deg \tilde{f}(E_1) = 2$ with counting multiplicity ($\because \#(E_1 \cdot \tilde{D}) = 2$).
See note P446.

$\Rightarrow$ By P174, $\tilde{f}(E_1)$ is a line in $\mathbb{P}^3$.

" Question: $(a_1\tilde{\sigma}_1 + a_2\tilde{\sigma}_2 = 0) \overset{?}{=} \tilde{D}$,
where $(\sigma_1 = 0) = D_1 \quad (\sigma_2 = 0) = D_2 \qquad D = (a_1\sigma_1 + a_2\sigma_2 = 0)$
and $\tilde{D}_1, \tilde{D}_2, \tilde{D}$ proper transforms of D_1, D_2, D respectively.

Let $\sigma = a_1\sigma_1 + a_2\sigma_2$.

$\Rightarrow$ Since $\pi : \tilde{\mathbb{P}}^2 \longrightarrow \mathbb{P}^2$ is one to one outside exceptional divisors, $\tilde{\sigma} = \tilde{\tau}$ outside exceptional divisors, $\tilde{\tau} = a_1\tilde{\sigma}_1 + a_2\tilde{\sigma}_2$.

$\Rightarrow \tilde{\sigma} = \tilde{\tau}$ by the identity theorem.

$\dim H^0(\mathbb{P}^1, \mathcal{O}(2H)) = \binom{1+2}{1} = 3 \quad \cdots (*)$

$$\begin{tikzcd} {[2H]} \arrow[r] \arrow[d] \arrow[rr, bend left] & {[H]} \arrow[d] & \\ E_1 \arrow[r, "\tilde{f}|"] & \mathbb{P}^3 & \end{tikzcd}$$
$E_1 = \mathbb{P}^1$

since $\#(\tilde{D} \cdot E_1) = 2$.

$\tilde{\sigma}_1|E_1 = \tilde{\sigma}_2|E_1 = \tilde{\sigma}_3|E_1$ since σ_i's have the same slopes ($\because \sigma_i = \Delta + \ell_i$.)