$\vdash$ By the Kodaira embedding theorem. $\exists\ f : M \to \mathbb{P}^N$ which is an embedding.

More presisely,

$$\iota_{L^k} : M \longrightarrow \mathbb{P}^N \text{ is an embedding.}$$

$\Rightarrow$ By p177. $L^k = \iota_{L^k}^*(H)$ is a positive l.b.

Let $L^k = \mathcal{L}$.

$$\Rightarrow \quad M \xrightarrow{\iota_{\mathcal{L}}} \mathbb{P}^N \text{ is an embedding}$$

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\tilde{\iota}_{\mathcal{L}}"] \arrow[d] & H \supset \tilde{\iota}_{\mathcal{L}}(\mathcal{L}) \arrow[d] \\
M \arrow[r, "\iota_{\mathcal{L}}"] & \mathbb{P}^N \supset \iota_{\mathcal{L}}(M)
\end{tikzcd}$$

$$\Rightarrow \quad \tilde{\iota}_{\mathcal{L}}(\mathcal{L}) = H\Big|_{\iota_{\mathcal{L}}(M)} \qquad \text{see p145.}$$

$\Rightarrow$ $\tilde{\iota}_{\mathcal{L}}(\mathcal{L})$ is the hyperplane bundle on $\iota_{\mathcal{L}}(M)$ (see also Milnor p26 lemma 3.1)

Thus we may assume that $M \subset \mathbb{P}^N$ an algebraic variety and $L \to M$ the hyperplane bundle.

Now $\iota_{E\otimes L^m}$ will be 1-1 if for all $x, y \in M$, the restriction map

$$(*) \qquad H^0(M, \mathcal{O}(E\otimes L^m)) \longrightarrow (E\otimes L^m)_x \oplus (E\otimes L^m)_y$$

is surjective.

$\vdash$ Let $\iota_K = f$, where $\iota_K = \iota_{L^K}$ & let $L^K = L$.