Before giving the inverse map, we need one remark. Given the data

$$\begin{cases} 0 \longrightarrow R \xrightarrow{i} S \xrightarrow{\pi} M \longrightarrow 0, \\ R \xrightarrow{j} N, \end{cases}$$

we may construct an extension

$$(F) \qquad 0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

as follows: Define $\mu = j \oplus i : R \longrightarrow N \oplus S$ and $F = (N \oplus S)/\mu(R)$. Then $n \oplus s \longmapsto -\pi(s)$ and $n \longmapsto n \oplus (0)$ gives the exact sequence (F).

Pf

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{\ell} & N \oplus S \big/ \mu(R) & \longrightarrow & M & \longrightarrow & 0 \\ & & \overset{\in}{n} & \longmapsto & n + \mu(R) & & \in & & \\ & & & & (n \oplus s) + \mu(R) & \longmapsto & -\pi(s) & & \end{array}$$

$\pi(\mu(r)) = \pi(j(r) \oplus i(r)) = \pi(i(r)) = 0$ by the exactness of $0 \to R \xrightarrow{i} S \xrightarrow{\pi} M \to 0$.

If $\pi((n \oplus s) + \mu(R)) = 0$, then $\underset{}{\overset{=0}{\pi(s)}}$

$\Rightarrow s = i(r)$.

$$\begin{aligned} \Rightarrow (n \oplus s) + \mu(R) &= (n \oplus i(r)) + \mu(R) = \\ &\quad ((n - j(r) + j(r)) \oplus i(r)) + \mu(R) \\ &= ((n - j(r)) \oplus (0)) + \mu(R) + (j(r) \oplus i(r)) + \mu(R) \\ &= ((n - j(r)) \oplus (0)) + \mu(R) + \mu(r) + \mu(R) \\ &= ((n - j(r)) \oplus (0)) + \mu(R) = \ell(n - j(r)) \end{aligned}$$