$\Rightarrow$ We have $\{(U_\alpha, f_\alpha + I)\} \in C^0(\underline{U}, \mathcal{O}_{(2)})$, $\mathcal{O}_{(2)} = \mathcal{O}/I^3$, so that $(\frac{f_\alpha}{f_\beta} + I) \in C^1(\underline{U}, \mathcal{O}^*_{(2)})$.

$\Rightarrow$ $(\frac{f_\alpha}{f_\beta} + I)$ gives an element in $H^1(L_{(2)}, \mathcal{O}^*_{(2)})$
I think $L_{(2)}$ is the nbd of L. ⌟

We now ask when there is an invertible sheaf $\mathcal{L} \in H^1(\mathcal{O}^*_{\mathbb{P}^2})$ that restricts to $\mathcal{L}_{(2)}$. For this we consider the exact sequence

$$0 \longrightarrow 1 + I^3 \longrightarrow \mathcal{O}^*_{\mathbb{P}^2} \longrightarrow \mathcal{O}^*_{(2)} \longrightarrow 0,$$

where $I \cong \mathcal{O}_{\mathbb{P}^2}(-L)$ is the ideal sheaf of the line $L \subset \mathbb{P}^2$ and $1 + I^3$ denotes the multiplicative sheaf of functions $1 + f$, where f vanishes to third order along L.

⌜ $\mathcal{O}_{\mathbb{P}^2}(-L)$ = sheaf of holomorphic functions on $\mathbb{P}^2$ vanishing to order ≥ 1 along L. $\Rightarrow$ $\mathcal{O}_{\mathbb{P}^2}(-L)$ = set of holomorphic functions on U vanishing to order ≥ 1 along $L \cap U$.
$\Rightarrow$ Clearly $\mathcal{O}_{\mathbb{P}^2}(-L) \cong I$. So $1 + I^3$ may be interpreted as the multiplicative sheaf of functions $1 + f$, where f vanishes up to order ≥ 3 along L. ⌟

Clearly $1 + I^3 \cong I^3$, and since $I^3 \cong \mathcal{O}_{\mathbb{P}^2}(-3L)$,

$H^1(\mathcal{O}_{\mathbb{P}^2}(-3L)) = 0$ by Kodaira vanishing,
$H^2(\mathcal{O}_{\mathbb{P}^2}(-3L)) \cong H^0(\mathcal{O}_{\mathbb{P}^2}) \cong \mathbb{C}$ by Kodaira-Serre duality,
$H^2(\mathcal{O}^*_{\mathbb{P}^2}) = 0$,

where the last step follows from the cohomology seque