P604. 4

" If $N \xrightarrow{\pi} M$ is any map of complex manifolds that is an isomorphism away from a point $x_0 \in M$, and such that the fiber of π at x_0 is isomorphic to $\mathbb{P}^{n-1}$, then $N \xrightarrow{\pi} M$ is the blow-up of M at x_0. "

① Δ disc in M, $\Delta \ni x_0$

$\pi^{-1}\Delta \xrightarrow{\pi} \Delta$

Suppose D is a divisor passing x_0 in Δ.

$\Rightarrow (\sigma = 0) = D \Rightarrow (\pi^*\sigma = 0)$ is a divisor in $\pi^{-1}\Delta$.

$\Rightarrow$ The formal sum $(\pi^*\sigma = 0) - \mathrm{mult}(\sigma)\cdot\pi^{-1}(x_0)$ is a divisor in $\pi^{-1}\Delta$.

② Given a smooth curve α passing x_0 in Δ, the curve is expressed as $(\sigma_1 = 0) \cap (\sigma_2 = 0) \cap \cdots \cap (\sigma_{n-1} = 0)$.

$\Rightarrow$ The "proper transform" $\tilde{\alpha}$ is expressed as

$$((\pi^*\sigma_1 = 0) - \pi^{-1}(x_0)) \cap ((\pi^*\sigma_2 = 0) - \pi^{-1}(x_0)) \cap \cdot ((\pi^*\sigma_{n-1} = 0) - \pi^{-1}(x_0))$$

which is one-dimensional variety in $\pi^{-1}\Delta$.

③ By ① & ②,

$\{\tilde{\alpha}(z)\} =$ closure of $\{\pi^{-1}(\alpha(z))\}_{z \neq 0}$, $\alpha(0) = x_0$

Clearly $\tilde{\alpha}(0) \in \pi^{-1}(x)$.

Thus we can define a map φ defined by

$$\begin{array}{ccc} \tilde{\Delta} & \longrightarrow & \pi^{-1}(\Delta) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ [x, \ell] & \longmapsto & \begin{cases} \pi^{-1}(x), & x \neq x_0 \\ \tilde{\alpha}(0), & x = x_0, \end{cases} \end{array}$$

where $[\alpha'(0)] = \ell$, $[\alpha'(0)] \in \mathbb{P}(T_x\Delta)$.