An obvious necessary condition is that (*) should hold locally. This has the following meaning: Let $p \in C \cap D$ and suppose that p is contained in the affine coordinate system $(x, y) \doteq [1, x, y]$. Then $f(x,y) = F(1, x, y)$, and $g(x,y) = G(1, x, y)$ generate an ideal I_p in the local ring $\mathcal{O}_p$ of germs of holomorphic functions defined in a nbd of p. The obvious necessary local condition is that, setting $h(x,y) = H(1, x, y)$,

$$(**) \qquad h(x,y) \in I_p \subset \mathcal{O}_p$$

for each $p \in C \cap D$. Conversely, we shall prove

Noether's AF+BG Theorem. If the local condition (**) are satisfied, then there is a global relation (*).

Proof. We let $I \subset \mathcal{O}$ be the sheaf of ideals generated by the various loc~~ations~~ (liza) f and g as above. Then I is coherent and $\operatorname{supp}(\mathcal{O}/I) = C \cap D$.

Γ By P697, I is coherent. $\operatorname{Supp}(\mathcal{O}/I) = \{ z \in \mathbb{P}^2 \mid I_z \neq \mathcal{O}_z \}$ $= C \cap D$ since $z \notin C \cap D \Rightarrow f \neq 0$ or $g \neq 0 \Rightarrow f$ or g is a unit. ⌟

Setting $d = m+k = n+\ell$ and $r = k-n = \ell-m$, the Koszul complex gives the global syzygy (cf. p. 698)

$$0 \to \mathcal{O}(r) \to \mathcal{O}(\ )_k \oplus \mathcal{O}(\ )_\ell \to I(d) \to 0,$$