The local representation (*) shows that R is an isomorphism. Q.E.D.

$\ulcorner$ To show R is isomorphic, we have only to prove that R_x is isomorphic,

$$R_x : \mathcal{H}^1(\Omega(*))_x \longrightarrow \bigoplus_{D \in \mathrm{Div} M} (\mathbb{C}_D)_x .$$

$$\varphi + d\Omega^0(*)(U) \longmapsto \bigoplus_i \lambda_i \cdot 1_{D_i}(x).$$

if $x \in D_i$, $\Rightarrow$ $D_i \subset (\varphi_\infty)$ in some nbd of x.

$\Rightarrow$ $R_x(\varphi + d\Omega^0(*)(U))$

$= \bigoplus_{D_i} \lambda_i$ where $(\varphi_\infty) = D_1 + \cdots + D_k$.

if $x \in D_j$ $\Rightarrow$ $1_{D_j}(x) = 0$.

$$\lambda_i = \frac{1}{2\pi\sqrt{-1}} \int_{\gamma_{D_i}} \varphi$$

If $R_x(\varphi + d\Omega^0(*)(U)) = 0$, φ has no residues in some nbd of x, and $\varphi = dg$, for some meromorphic function on the nbd of x. $\Rightarrow$ $\varphi \in d\Omega^0(*)(W_x)$, $W_x \subset U$.
$\Rightarrow$ $\varphi + d\Omega^0(*)(U) = 0$. $\Rightarrow$ R_x is one to one.
Given $\bigoplus_i \lambda_i 1_{D_i}(x)$, then only finite # of $1_{D_i}(x)$'s are non zero. $\Rightarrow$ Say $D_1, D_2, \cdots D_k$, $\Rightarrow$ Let $f_1 \cdots f_k$ be local defining function for $D_1, \cdots D_k$.
$\Rightarrow$

Let $\varphi = \sum \lambda_i \dfrac{df_i}{f_i}$. $\Rightarrow$ $R_x(\varphi + d\Omega^0(*)(U)) = \bigoplus_i \lambda_i 1_{D_i}(x)$.

Make-up explanation. $\lambda \in (\mathbb{C}_D)_x \cong \mathbb{C}$. $x \in D$.
$\Rightarrow$ $D = D_1 + \cdots + D_k$ D_i's irreducible at x.