Date

$\Rightarrow \exists$ not all zero a_i s.t $\sum a_i f_i$ is holoph. $\Rightarrow$
$\sum a_i f_i$ = constant = c $\Rightarrow$ $\sum a_i f_i^{i\text{ odd}} - c f_0 = 0$
$\Rightarrow$ $\{$ old f_i's $\}$ is a linearly dependent set. *.))

Equivalently, if z_i is a local coordinate on S centered around p_i and if we write

$$f_j(z_i) = a_{j,i}\, z_i^{-1} + \cdots\cdots$$

then the matrix $\begin{bmatrix} a_{1,1} & \cdots & a_{1d} \\ \vdots & & \vdots \\ a_{r1}, & \cdot & a_{rd} \end{bmatrix}$

has maximal rank r.

Γ We assume that p_i's are distinct.
For $r = 2$, $1 \le i \le 3$,

$$f_1(z_1) = a_{11} z_1^{-1} + \cdots \quad f_2(z_1) = a_{21} z_1^{-1} + \cdots$$
$$f_1(z_2) = a_{12} z_2^{-1} + \cdots \quad f_2(z_2) = a_{22} z_2^{-1} + \cdots$$
$$f_1(z_3) = a_{13} z_3^{-1} + \cdots \quad f_2(z_3) = a_{23} z_3^{-1} + \cdots$$

$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}, & a_{22}, & a_{23} \end{bmatrix}$ has rank 2. $\Leftrightarrow \exists$ no c_1, c_2

s.t, not all c_i's zero, $c_1(a_{11}, a_{12}, a_{13}) + c_2(a_{21}, a_{22}, a_{23}) = 0$. $\Leftrightarrow \exists$ no c_1, c_2 s.t $c_1 f_1 + c_2 f_2$ is holomorphic, not all c's zero.
For $r = 3$, $1 \le i \le 2$,

$$f_1(z_1) = a_{11} z_1^{-1} + \cdots \quad f_1(z_2) = a_{12} z_2^{-1} + \cdots$$
$$f_2(z_1) = a_{21} z_1^{-1} + \cdots \quad f_2(z_2) = a_{22} z_2^{-1} + \cdots$$

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