$v_{12}, v_{13}) + g_{22}(v_{21}, v_{22}, v_{23}))$ span $\langle (v_{11}, v_{12}, v_{13}), (v_{21}, v_{22}, v_{23}) \rangle$.
Thus this implies that the map ι is independent of the choice of a local frame for E.

($\Rightarrow$) Suppose $\sigma_1, \sigma_2, \cdots \sigma_{k-r+1}$ are linearly independent at x. Choose $\sigma_1, \sigma_2, \cdots \sigma_{k-r+1}, \omega_{k-r+2}, \cdots \omega_k$ as a local frame for E near x.

$$\Rightarrow \quad \begin{aligned} \sigma_1 &= \sigma_1 \\ \sigma_2 &= 0 + \sigma_2 + 0 \cdots \\ \sigma_3 &= \qquad\qquad \sigma_3 \\ &\vdots \\ \sigma_{k-r+1} &= \cdots \qquad\qquad \sigma_{k-r+1}. \\ &\vdots \\ \sigma_n &= * \sigma_1 + * \sigma_2 + \cdots \end{aligned}$$

$$\Rightarrow \iota(x) = \left\langle \begin{array}{l} \overbrace{(1, 0, \cdots 0\ 0}^{k-r+1}\ *, * \cdots *) = W_1 \\ (0, 1, \cdots 0\ 0\ *\ * \cdots *) = \\ \ \vdots \quad \vdots \qquad \vdots\ \ \vdots \qquad\qquad \vdots \\ (0 \cdots\cdots 0\ 1\ *\ * \cdots *) = W_{k-r+1} \\ (* \quad \cdots \qquad\qquad *) \end{array} \right\rangle$$

$$\dim(\Lambda + V_{n-k+r-1}) = \dim \Lambda + \dim V_{n-k+r-1} - \dim(\Lambda \cap V_-)$$

$$\Rightarrow \dim(\Lambda \cap V_{n-k+r-1}) = k + n-k+r-1 - \dim(\Lambda + V_-) \cdots \circledast$$

$$\geq \quad n+r-1-n = r-1, \text{ since } \dim(\Lambda + V_-) \leq n.$$

$\Rightarrow \{W_1, W_2, \cdots W_{k-r+1}, e_{k-r+2}, \cdots e_n\}$ is a set of linearly independent vectors. $\Rightarrow \dim(\Lambda + V_{n-k+r-1}) = n$.
$\Rightarrow$ This contradicts to the assumption that $\dim(\Lambda \cap V_{n-k+r-1}) \geq r$, since by $\circledast$ $\dim(\Lambda \cap V_{n-k+r-1}) = r-1$.
Thus $\sigma_1, \cdots \sigma_{k-r+1}$ must be linearly dependent.