invariant of a symmetric quadratic form over $\mathbb{C}$ is its rank, two quadrics $F, F' \subset \mathbb{P}^n$ will be projectively isomorphic if and only if they have the same rank.

$\ulcorner$ See note P732

Now taking partials, $\frac{\partial}{\partial X_i} Q(X, X) = 2 \sum_j q_{ij} X_j$,

we deduce that the singular locus of F is just the linear subspace of $\mathbb{P}^n$ corresponding to the kernel of the matrix Q on $\mathbb{C}^{n+1}$; thus

A quadric $F \subset \mathbb{P}^n$ is smooth if and only if it has maximal rank $n+1$,

and more generally,

A quadric $F \subset \mathbb{P}^n$ of rank $n-k$ is singular along a k-plane $\Lambda \subset F \subset \mathbb{P}^n$.

$\ulcorner$ $\{ \nabla Q = 0 \} = \{ \frac{\partial Q(X, X)}{\partial X_i} = 0 \text{ for all } i \}$

$= \{ X \mid QX = 0 \}$, $\quad Q : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$

$$X = \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix} \longmapsto QX$$