$\Rightarrow (R_{g^{-1}}^* \sigma)\left(g\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = (a\,dy + b\,dx)\left(v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}\right)$
$= a v_2 + b v_1$

$(R_{g^{-1}}^* \sigma)\left(g\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = (a'\,dy + b'\,dx)\left((a_{11}v_1 + a_{12}v_2)\frac{\partial}{\partial x} + (a_{21}v_1 + a_{22}v_2)\frac{\partial}{\partial y}\right) = b'(a_{11}v_1 + a_{12}v_2) + a'(a_{21}v_1 + a_{22}v_2)$
$= v_1(a_{11}b' + a_{21}a') + v_2(a_{12}b' + a_{22}a')$.

$\Rightarrow \quad b = a_{11}b' + a_{21}a'$
$\quad\;\; a = a_{12}b' + a_{22}a'$. $\Rightarrow \begin{pmatrix} a_{11}, & a_{21} \\ a_{12}, & a_{22} \end{pmatrix}\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$.

Note that we didn't use either the properness or orthogonality.

$\Rightarrow$ If σ is invariant under the proper orthogonal group, $\sigma = (a_{11}x + a_{12}y)\,dx + (a_{21}x + a_{22}y)\,dy$.

$\Rightarrow$ If $d\sigma = 0$, $a_{12} + a_{21} = 0$. $\Rightarrow \sigma = a_{11}x\,dx + a_{12}(y\,dx - x\,dy) + a_{22}y\,dy$.

Forget the argument above.

Suppose τ is invariant on S^n under the proper orthogonal group. At x_0, $\tau(e_1, e_2, \cdots e_n) = \alpha\,\sigma(e_1, \cdots e_n)$ for some constant α.

$\Rightarrow R_g^*\tau(e_1 \cdots e_n) = \tau(e_1, \cdots e_n) = \alpha\,\sigma(e_1 \cdots e_n)$
$= \alpha R_g^*\sigma(e_1 \cdots e_n)$. $\Rightarrow \tau = \alpha\sigma$ since $\tau(v_1 \cdots v_n)$ is dependent on the value of $\tau(e_1, e_2, \cdots e_n)$

Thus σ is the unique form on S^n which is invariant under the proper orthogonal group and that has integral 1 over S^n. ⌟