Lemma. Given $h(z,w) \in \mathcal{O}$. if $\mathrm{Ord}_c(h) \geq \mathrm{Ord}_c(g)$, then $h \in I$.

Proof. According to the local duality theorem we must prove that

$$\mathrm{Res}_{\{0\}}\left(\frac{hk\,dz\wedge dw}{fg}\right) = 0$$

for all $k \in \mathcal{O}$.

See p 659

Since $\mathrm{Ord}_c(hk) \geq \mathrm{Ord}_c(h)$ this will follow from showing that

$$\mathrm{Res}_{\{0\}}\left(\frac{h\,dz\wedge dw}{fg}\right) = 0$$

whenever $\mathrm{Ord}_c(h) \geq \mathrm{Ord}_c(g)$.

$\mathrm{Ord}_c\left(\frac{h}{fg}\right) \geq -1$. $\Rightarrow$ If $\mathrm{Res}_{\{0\}}\left(\frac{h\,dz\wedge dw}{fg}\right) = 0$, then

$$\mathrm{Res}_{\{0\}}\left(\frac{hk\,dz\wedge dw}{fg}\right) = 0 .$$

We may choose local coordinates so that $f(z,w) = z$. Then, by iteration of the residue integral

$$\mathrm{Res}_{\{0\}}\left(\frac{h\,dz\wedge dw}{fg}\right) = \frac{1}{2\pi\sqrt{-1}}\int_{|g|=\epsilon}\left(\int_{|z|=\epsilon}\frac{h(z,w)\,dz}{g(z,w)\,z}\right)dw$$

$$= \frac{1}{2\pi\sqrt{-1}}\int_{|g(0,w)|=\epsilon}\frac{h(0,w)}{g(0,w)}\,dw$$