Choose coordinates $(z_1, z_2 \cdots z_n)$ s.t $f(z) = z_n$.

$\Rightarrow$ RHS $= \int_{Z^*} \varphi = 0$ unless $\varphi = \alpha(z)\, dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$.

$\Rightarrow$ LHS $= \frac{\sqrt{-1}}{\pi} \int_{\mathbb{C}^n} \log|z_n|\, \bar{\partial}\partial\varphi = ?$ For simplicity, let $n = 3$

$\varphi = \alpha(z)\, dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3 \Rightarrow \bar{\partial}\partial\varphi = \frac{\partial^2 \alpha}{\partial z_3 \partial \bar{z}_2}\, dz_1 \wedge dz_2 \wedge \cdots \wedge d\bar{z}_3$

$\Rightarrow \frac{\sqrt{-1}}{\pi} \int_{\mathbb{C}^3} \log|z_3| \frac{\partial^2 \alpha}{\partial z_3 \partial \bar{z}_2}\, dz \wedge d\bar{z} = \frac{\sqrt{-1}}{\pi} \int_{\mathbb{C}^3} \log|z_3| \frac{\partial g}{\partial \bar{z}_2}\, dz \wedge d\bar{z}$.

where $g = \frac{\partial \alpha}{\partial z_3}$, g has compact support

$= \frac{\sqrt{-1}}{\pi} \int_{\mathbb{C}^3} \log|z_3| \overset{\frac{\partial g}{\partial \bar{z}_2}}{\vee}\, dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3$

$= \pm \frac{\sqrt{-1}}{\pi} \int_{\mathbb{C}^2} \left(\int_{\mathbb{C}} \log|z_3| \overset{\frac{\partial g}{\partial \bar{z}_2}}{\vee}\, dz_2 \wedge d\bar{z}_2 \right) \wedge dz_1 \wedge d\bar{z}_1 \wedge dz_3 \wedge d\bar{z}_3$

Consider $\frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{\partial g}{\partial \bar{z}_2}\, dz_2 \wedge d\bar{z}_2 = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \frac{1}{z_2} \frac{\partial z_2 g}{\partial \bar{z}_2}\, dz_2 \wedge d\bar{z}_2$

$= z_2 g(z)\big|_{z_2 = 0} = 0 \Rightarrow$ LHS $= 0$.

Thus we may assume that $\varphi(z) = \alpha(z)\, dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$

RHS $= \int_{\mathbb{C}^{n-1}} \alpha(z_1, z_2, \cdots, 0)\, dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$

LHS $= \frac{\sqrt{-1}}{\pi} \int_{\mathbb{C}^n} \log|z_n| \bar{\partial}\partial\varphi = \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{C}^n} \log|z_n|^2 \frac{\partial^2 \alpha}{\partial \bar{z}_n \partial z_n}\, dz \wedge d\bar{z}$

$= \frac{\sqrt{-1}}{2\pi} \int_{\mathbb{C}^{n-1}} \left(\int_{\mathbb{C}} \log|z_n|^2 \frac{\partial^2 \alpha}{\partial \bar{z}_n \partial z_n}\, dz_n \wedge d\bar{z}_n \right) \wedge dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \cdots$

$= \int_{\mathbb{C}^{n-1}} \alpha(z_1, \cdots z_{n-1}, 0)\, dz_1 \wedge \cdots \wedge dz_{n-1} \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_{n-1}$ by the