We have a "wedge product"

$$\wedge : A^{p,q}(E) \otimes A^{p',q'}(E^*) \longrightarrow A^{p+p',q+q'}(M)$$

defined by

$$(\eta \otimes s) \wedge (\eta' \otimes s') = \langle s, s' \rangle \cdot \eta \wedge \eta';$$

we define an operator

$$*_E : A^{p,q}(E) \longrightarrow A^{n-p,n-q}(E^*)$$

by requiring, for $\eta, \psi \in A^{p,q}(E)$,

$$(\eta, \psi) = \int_M \eta \wedge *_E \psi .$$

⟦ $\eta \otimes s \in A^{p,q}(E)$, where η is (p,q)-form and s is a section of E. ⟧

Explicitly, if $\{e_\alpha\}$ and $\{e_\alpha^*\}$ are dual unitary frames for E and E^*, then for $\eta \in A^{p,q}(E)$ written as

$$\eta = \sum \eta_\alpha \otimes e_\alpha, \quad \eta_\alpha \in A^{p,q}(M),$$

$$*_E \eta = \sum * \eta_\alpha \otimes e_\alpha^*,$$

where $*$ is the usual star operator on $A^{p,q}(M)$.

⟦ $\eta = \sum \eta_\alpha \otimes e_\alpha \qquad \psi = \sum \psi_\alpha \otimes e_\alpha$

$*_E \psi = \sum * \psi_\alpha \otimes e_\alpha^* \quad \Rightarrow \quad \eta \wedge *_E \psi = \sum \eta_\alpha \wedge * \psi_\alpha$

$$\int \eta \wedge *_E \psi = \int \sum_\alpha \eta_\alpha \wedge * \psi_\alpha = \sum_\alpha \int \eta_\alpha \wedge * \psi_\alpha = \sum_\alpha \int (\eta_\alpha, \psi_\alpha) \Phi$$

$$= \sum_\alpha \langle \eta_\alpha, \psi_\alpha \rangle = \langle \eta, \psi \rangle = \sum_\alpha \langle \eta_\alpha, \psi_\alpha \rangle$$