Pu_ϵ. $\quad Pu_\epsilon = P(u_\epsilon) = P(u_\epsilon) - (Pu)_\epsilon + (Pu)_\epsilon$.

$$(Pu)_\epsilon(x) = \int_{\mathbb{R}^n} (Pu)(y)\, \chi_\epsilon(x-y)\, dy$$

$$= \int_{\mathbb{R}^n} v(y)\, \chi_\epsilon(x-y)\, dy = v_\epsilon(x)$$

$$\Rightarrow \quad \|(Pu)_\epsilon\|_s = \|v_\epsilon\|_s .$$

For constant-coefficient operators this is zero, and so in general we may expect a bound in terms of the s-norm of u and 1-norm of the $a_{ij}{}^k(x)$'s. For simplicity we do the case $s=0$, the general argument being the same.

$$(Pu_\epsilon) - (Pu)_\epsilon$$

$$= \Delta u_\epsilon + \sum a_k(x) \frac{\partial u_\epsilon(x)}{\partial x_k} + b(x) u_\epsilon(x)$$

$$- \int_{\mathbb{R}^n} (Pu)(y)\, \chi_\epsilon(x-y)\, dy = \Delta u_\epsilon(x) + \sum a_k(x) \frac{\partial u_\epsilon(x)}{\partial x_k}$$

$$+ b(x)\, u_\epsilon(x) - \int_{\mathbb{R}^n} v(y)\, \chi_\epsilon(x-y)\, dy$$

$$= \int_{\mathbb{R}^n} u(y)\, \Delta_x \chi_\epsilon(x-y)\, dy + \sum a_k(x) \int_{\mathbb{R}^n} u(y) \frac{\partial}{\partial x_k} \chi_\epsilon(x-y)\, dy$$

$$+ b(x) \int_{\mathbb{R}^n} u(y)\, \chi_\epsilon(x-y)\, dy - \int_{\mathbb{R}^n} u(y)\, P_y\, \chi_\epsilon(x-y)\, dy$$