

But, by the adjunction formula,

$$\pi(D) = \frac{L \cdot L + L \cdot K}{2} + 1;$$

thus we have

$$\chi(L) = \chi(\mathcal{O}_M) + \frac{L \cdot L - L \cdot K}{2}.$$

$$\begin{aligned} \text{If } \chi(\mathcal{O}_D(L)) &= -\chi(\mathcal{O}_M) + \chi(L) \\ &= -\pi(D) + L \cdot L + 1 \\ &= -\frac{L \cdot L + L \cdot K}{2} - 1 + L \cdot L + 1 \\ &= \frac{L \cdot L - L \cdot K}{2} \Rightarrow \chi(L) = \chi(\mathcal{O}_M) + \frac{L \cdot L - L \cdot K}{2} \quad \square \end{aligned}$$

This formula holds for an arbitrary line bundle L on M . We just choose a divisor D on M sufficiently positive so that both the linear series $|D|$ and $|L+D|$ contain smooth, irreducible divisors; setting

$$L' = L + D$$

the exact sequence

$$0 \rightarrow \mathcal{O}_M(L) \rightarrow \mathcal{O}_M(L') \rightarrow \mathcal{O}_D(L') \rightarrow 0$$

gives

$$\chi(L) = \chi(L') - \chi(L'|_D).$$

If Choose a divisor D on M so that $\iota_{[L+D]}: M \rightarrow \mathbb{P}^N$ and $\iota_{[D]}: M \rightarrow \mathbb{P}^M$ are embeddings. This is possible because of Kodaira embedding theorem P181.