Moreover, $\bar\partial \log \bar f_\alpha = \overline{\partial \log f_\alpha}$ and, since ψ is real,

this implies

$$\int_{\partial D(\epsilon)} \bar\partial \log \bar f_\alpha \wedge \psi = \overline{\int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi} .$$

Thus in U_α,

$$\lim_{\epsilon\to 0} \frac{2\pi}{i} \int_{\partial D(\epsilon)} d^c \log |S|^2 \wedge \psi = \lim_{\epsilon\to 0} -i \,\mathrm{Im} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi .$$

$\ulcorner$ $$\frac{2\pi}{i} \int_{\partial D(\epsilon)} d^c \log |S|^2 \wedge \psi = \frac{2\pi}{i} \frac{i}{4\pi} \int_{\partial D(\epsilon)} \bar\partial \log \bar f_\alpha \wedge \psi - \frac{2\pi}{i}\frac{i}{4\pi} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi$$

$$+ \frac{2\pi}{i} \int_{\partial D(\epsilon)} d^c \log h_\alpha \wedge \psi$$

Taking the limit $\epsilon \to 0$, and $\bar\partial \log \bar f_\alpha = \overline{\partial \log f_\alpha}$,

$$\lim_{\epsilon\to 0} \frac{2\pi}{i} \int_{\partial D(\epsilon)} d^c \log |S|^2 \wedge \psi = \frac{1}{2} \overline{\int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi} - \frac{1}{2} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi$$

$$= \frac{1}{2} \lim_{\epsilon\to 0} \left(\overline{\int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi} - \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi \right)$$

$$= -i \lim_{\epsilon\to 0} \mathrm{Im} \int_{\partial D(\epsilon)} \partial \log f_\alpha \wedge \psi$$ $\lrcorner$

Now in the nbd of any smooth point $z_0 \in V \cap U_\alpha$, we can find a holomorphic coordinate system $w = (w_1, w_2, \cdots w_n)$ with $w_1 = f_\alpha$.

$\ulcorner$ Since f_α is a local defining function on U_α,
$Df_\alpha(z_0) \neq 0$. $\lrcorner$