Hodge Decomposition. For a compact Kähler manifold M, the complex cohomology satisfies.

$$\begin{cases} H^r(M, \mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(M). \\ H^{p,q}(M) = \overline{H^{q,p}(M)}. \end{cases}$$

Γ $H^r(M, \mathbb{C}) \overset{\text{de Rham iso.}}{=} H^r_{DR}(M, \mathbb{C}) = \mathcal{H}_d^r(M) = \bigoplus_{p+q=r} \mathcal{H}_d^{p,q}(M)$

$$= \bigoplus_{p+q=r} H^{p,q}(M)$$

$$H^{p,q}(M) = \mathcal{H}_d^{p,q}(M) \cong \overline{\mathcal{H}_d^{q,p}(M)} = \overline{H^{q,p}(M)} \quad \lrcorner$$

Since $\Delta_d = 2\Delta_{\bar\partial}$, we have $\mathcal{H}_d^{p,q}(M) = \mathcal{H}_{\bar\partial}^{p,q}(M)$ and consequently

$$H^{p,q}(M) \cong \mathcal{H}_d^{p,q}(M) \cong \mathcal{H}_{\bar\partial}^{p,q}(M) \underset{\text{Dolbeault Iso}}{\cong} H^q(M, \Omega^p).$$

In particular, taking $q=0$,

$H^{p,0}(M) = H^0(M, \Omega^p)$ is the space of holomorphic p-forms. The holomorphic forms are therefore harmonic for any Kähler metric on a compact manifold.

We note also that

The Betti numbers $b_{2q+1}(M)$ of odd degree are even.

pf). If we define the Hodge numbers by

$$h^{p,q}(M) = \dim H^{p,q}(M),$$

then the Hodge decomposition gives

$$b_r(M) = \sum_{p+q=r} h^{p,q}(M) \qquad h^{p,q}(M) = h^{q,p}.$$

Taking $r = 2q+1$, we find