Comment : ($\Rightarrow$) It is easy.
($\Leftarrow$) To prove the other direction, we needed some arrangement, so we prove

$$Θ = \partial\bar{\partial}\rho + Θ' \qquad e^{\rho}|s|^2 = |s|'^2$$

$$\Rightarrow \left[\frac{i}{2\pi}Θ\right] = \left[\frac{i}{2\pi}Θ'\right].$$

proof). Let G_d denote the Green's operator associated to the Laplacian Δ_d, and similarly for G_∂ and $G_{\bar\partial}$. From the basic identity of p113

$$\frac{1}{2}\Delta_d = \Delta_\partial = \Delta_{\bar\partial}$$

it follows first that

$$2G_d = G_\partial = G_{\bar\partial},$$

and then that all the operator $d, \partial, \bar\partial, d^*, \partial^*$ and $\bar\partial^*$ commute with the Green's operators.

Γ Since $I = H + \Delta_d G_d$, $I = H + \Delta_{\bar\partial} G_{\bar\partial}$ & $I = H + \Delta_\partial G_\partial$, where H is the projection to the harmonic space,

$$\Delta_d G_d = \Delta_{\bar\partial} G_{\bar\partial} = \Delta_\partial G_\partial.$$

$\Rightarrow$ $2\Delta_\partial G_d = \Delta_\partial G_\partial$ $\Rightarrow$ $\Delta_\partial(2G_d) = \Delta_\partial G_\partial$
$\Rightarrow$ By the uniqueness, $2G_d = G_\partial$.

Similarly, we get $2G_d = G_\partial = G_{\bar\partial}$. $\lrcorner$ The rest follows trivially from the commutativity.