We make one final remark about sections of line bundles, which will be used repeatedly throughout the book. Recall that if $D = \sum a_i V_i$ is any effective divisor on the complex manifold M, $s_0 \in H^0(M, \mathcal{O}([D]))$ a section of $[D]$ with divisor D, then tensoring with s_0 gives an identification between the meromorphic functions on M with poles of order $\leq a_i$ on V_i and holomorphic sections of $[D]$.

More generally, if E is any holomorphic vector bundle on M, $\mathcal{E}$ its sheaf of holomorphic sections, we write $\mathcal{E}(D)$ for the sheaf of meromorphic sections of E with poles of order $\leq a_i$ on V_i, $\mathcal{E}(-D)$ for the sheaf of sections of E vanishing to order $\geq a_i$ along V_i.

Again, tensoring with s_0 or s_0^{-1} gives identifications

$$\mathcal{E}(D) \xrightarrow{\otimes s_0} \mathcal{O}(E \otimes [D]),$$

(*)

$$\mathcal{E}(-D) \xrightarrow{\otimes s_0^{-1}} \mathcal{O}(E \otimes [-D]).$$

$\sqcap$ $\quad \mathcal{E}(D) \ni s \quad \Rightarrow \quad$ On V_i, $\operatorname{ord}_{V_i}(s) \geq 0$ or

if $\operatorname{ord}_{V_i}(s) < 0$, $\quad \operatorname{ord}_{V_i}(s) \leq a_i$.

$s \otimes s_0 \in \mathcal{O}(E \otimes_{\mathcal{O}} [D])$ since. $s_\alpha \otimes_{\mathcal{O}} (s_0)_\alpha$ on U_α.
$(s_0)_\alpha = g_\alpha^{a_i} h$ where g_α is a local defining function on V_i.

$\Rightarrow \quad s_\alpha \otimes_{\mathcal{O}} (s_0)_\alpha = s_\alpha \otimes g_\alpha^{a_i} h = s_\alpha g_\alpha^{a_i} \otimes h$ is holomorphic

($\because$ $s_\alpha g_\alpha^{a_i}$ is holomorphic since s_α has a pole of order $\leq a_i$)

Given any $\sigma \in \mathcal{O}(E \otimes [D])$, $\sigma_\alpha \in \mathcal{O}(E \otimes [D])(U_\alpha)$.
$\Rightarrow \quad \sigma_\alpha = \sum \phi_{\alpha i} \otimes h_{\alpha i}$. where $\phi_{\alpha . i} \in \Gamma(E|U_\alpha)$. $h_{\alpha . i} \in \Gamma([D]|U_\alpha)$