$$|S_R - S_{R'}| \leq \sum_{\|\xi\| \geq R} |u_\xi| = \sum_{\|\xi\| \geq R} \frac{\left((1+\|\xi\|^2)^{[\frac{n}{2}]+1} |u_\xi|^2\right)^{\frac{1}{2}}}{\left((1+\|\xi\|^2)^{[\frac{n}{2}]+1}\right)^{\frac{1}{2}}}$$

$$\leq \|u\|_{[\frac{n}{2}]+1} \left[\sum_{\|\xi\| \geq R} \left(\frac{1}{(1+\|\xi\|^2)^{[\frac{n}{2}]+1}}\right)^{\frac{1}{2}} \right]$$ since $(1+\|\xi\|^2)^{[\frac{n}{2}]+1} |u_\xi|^2 \leq \left(\|u\|_{[\frac{n}{2}]+1}\right)^2$.

Now apply the integral test in $\mathbb{R}^n$ to conclude that

$$\sum_{\xi} \left(\frac{1}{(1+\|\xi\|^2)^{[\frac{n}{2}]+1}}\right)^{\frac{1}{2}} \leq \sum_{\xi \neq 0} \frac{1}{\|\xi\|^{n+1}} \leq \int_N^\infty \frac{n}{x^{n+1}}\, dx.$$

converges, from which it follows that $S_R(x)$ converges uniformly to $\varphi \in C^0(T)$ with $\varphi_\xi = u_\xi$.

⌈ $|S_R(x) - S_{R'}(x)| \leq f(R)$. $R' \geq R$. $\lim_{R \to \infty} f(R) = 0$.

$\{S_R(x)\}_R$ is a sequence of function. indexed by R.

$\Rightarrow$ Given $\epsilon > 0$, $\exists$ R_0 s.t if $R, R' \geq R_0$

$$|S_R(x) - S_{R'}(x)| < \epsilon \quad \text{for all } x.$$

$\Rightarrow$ Take $R' \longrightarrow \infty$, $\Rightarrow$ $S_{R'}(x) = u$.

$\Rightarrow$ $S_R \longrightarrow u$ converses uniformly.

$\Rightarrow$ u is continuous. $\Rightarrow$ we are done ⌋

Now we proceed by induction on s. Since the proof for general n involves only inessential formalism beyond what we have just done together with the one-variable case, we shall complete the argument only when $n=1$.

So, we suppose $H_{s+1} \subset C^s(T)$ and $u = \sum_{\xi \in \mathbb{Z}} u_\xi e^{i\xi x}$ satisfies $u \in H_{s+2}$, i.e