$X_1, \cdots X_n)$ be linear coordinates on $\mathbb{C}^{n+1}$, π_* and $\mathcal{E}$ as in Section 3 of this chapter, and $(\alpha_0, \cdots \alpha_n)$ any $(n+1)$-vector of distinct nonzero complex numbers. Consider the vector field on $\mathbb{P}^n$,

$$v(X) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i}$$

(since $\pi_* \sum X_i (\partial/\partial X_i) \equiv 0$, we may as well take $\sum \alpha_i = 0$).

⌈ See P409. $\sum X_i \, \pi_*(\frac{\partial}{\partial X_i}) = 0$.

$$v(X) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i}$$

$$= \sum_{i=0}^{n} \alpha_i X_i \pi_* \frac{\partial}{\partial X_i} = \alpha_0 X_0 \pi_* \frac{\partial}{\partial X_0} + \sum_{i=1}^{n} \alpha_i X_i \pi_* \frac{\partial}{\partial X_i}$$

$$= \alpha_0 \left(- \sum_{i=1}^{n} X_i \pi_* \frac{\partial}{\partial X_i}\right) + \sum_{i=1}^{n} \alpha_i X_i \pi_* \frac{\partial}{\partial X_i}$$

$$= \sum_{i=0}^{n} (\alpha_i - \alpha_0) X_i \pi_* \frac{\partial}{\partial X_i}$$

If $\sum \alpha_i = 0$, then

$$v(X) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i} = \alpha_0 X_0 \pi_* \frac{\partial}{\partial X_0} + \sum_{i=1}^{n} \alpha_i X_i \pi_* \frac{\partial}{\partial X_i}$$

$$= - \sum_{i=1}^{n} \alpha_i X_0 \pi_* \frac{\partial}{\partial X_0} + \sum_{i=1}^{n} \alpha_i X_i \pi_* \frac{\partial}{\partial X_i}$$

$$= \sum_{i=1}^{n} \left(X_i \pi_* \frac{\partial}{\partial X_i} - X_0 \pi_* \frac{\partial}{\partial X_0}\right) \alpha_i .$$

So what ? Nonsense. No! see P406 ⌋

As we have seen, v vanishes exactly at the point $p_i = [0, \cdots \underset{i}{1} \cdots 0]$; in terms of Euclidean coordinates

$$x_j = \frac{X_j}{X_i}, \quad j \neq i,$$