Remark: $Q = E/S$ is a holomorphic v.b.

Q is C^∞- bundle isomorphic to $S^\perp$.

$\Rightarrow$ $D_Q : \mathcal{A}^0(Q) \longrightarrow \mathcal{A}^1(Q)$

$\Rightarrow$ ~~By~~ Through the bundle isomorphism, $\exists\ D : \mathcal{A}^0(S^\perp) \longrightarrow \mathcal{A}^1(S^\perp)$.

$$\mathcal{A}^0(S^\perp) \ni \sigma \qquad \begin{tikzcd} M \arrow[r, "\sigma"] \arrow[dr, dashed, "f\circ\sigma"'] & S^\perp \arrow[d, leftrightarrow, "f"] \\ & Q \end{tikzcd} \qquad \begin{tikzcd} T^*M \otimes Q \arrow[d, leftrightarrow, "1\otimes f \equiv f"] \\ T^*M \otimes S^\perp \end{tikzcd}$$

convention

Define $D\sigma = f^{-1}\circ D_Q(f\circ\sigma)$

$$D(h\sigma) = f^{-1}\circ D_Q(f\circ(h\sigma)) = f^{-1}\circ D_Q(h\, f(\sigma))$$

$$= f^{-1}(dh \otimes f(\sigma) + h\, D_Q(f(\sigma)))$$

$$= dh \otimes f^{-1}(f\sigma) + h\, f^{-1}D_Q(f(\sigma)) = dh\otimes\sigma + h\, D\sigma$$

Another basic fact that comes out of this calculation is the following:

$E \longrightarrow M$ holomorphic bundle.

If $\exists$ global holomorphic sections $\sigma_1, \sigma_2 \cdots \sigma_n \in \Gamma(M, E)$, s.t for all $x \in M$, $\{\sigma_1(x) \cdots \sigma_n(x)\}$ generates E_x.

$\Rightarrow$ $\exists$ surjective holomorphic bundle map $f : M\times\mathbb{C}^n \longrightarrow E \to 0$ given by

$$(x, \lambda) \longmapsto \sum \lambda_i \sigma_i(x) \in E_x \quad \text{for all } x\in M,\ \lambda\in\mathbb{C}^n.$$

It follows that, if we give E the metric induced from the Euclidean metric on $M\times\mathbb{C}^n$, $\Theta_E \geq 0$.