The spectral decomposition for G on $L^2(T)$ is just Fourier series.

⎡⎡ $G(e^{i\langle \xi, x\rangle}) = \frac{1}{\|\xi\|^2} e^{i\langle \xi, x\rangle}$ for $\xi \neq 0$.

$e^{i\langle \xi, x\rangle}$ is an eigenfunction for G whose eigenvalue is $\|\xi\|^2$. ⌋⌋

At this juncture, the observations of the preceding paragraph more than establish the Hodge theorem for zero-forms on a torus. The essential point is this:

The operator $I + \Delta_d : H_s \longrightarrow H_{s-2}$ is trivially bounded, since Δ is second order. More importantly, the identity $\|(I+\Delta_d)\varphi\|^2_{s-2} = \|\varphi\|^2_s$. allows us to

invert $I + \Delta_d$ on $L^2(T)$ using the closed graph theorem.

⎡⎡ $\|(I+\Delta_d)\varphi\|^2_{s-2} = \sum (1+\|\xi\|^2)^{s-2} |((I+\Delta_d)\varphi)_\xi|^2$.

but since $((I+\Delta_d)\varphi)_\xi = \varphi_\xi + \|\xi\|^2 \varphi_\xi = (1+\|\xi\|^2)\varphi_\xi$,

$= \sum (1+\|\xi\|^2)^s |\varphi_\xi|^2 = \|\varphi\|^2_s$ ⌋⌋

$I + \Delta_d$ has a closed image $\Rightarrow$ By P294. $\exists$ an orthogonal complement. (See Rudin Functional Analysis)
This inverse is a compact smoothing operator and contains the information of the Green's operator.
If, on a general compact manifold M, we carry over the Sobolev-space formalism and can prove the basic estimate $\|(I+\Delta_d)\varphi\|^2_{s-2} \geq C_s \|\varphi\|^2_s$ by calculus,