$$\sigma_{\alpha_0,\cdots\alpha_p} \in \mathbb{Z}(\cap St(v_{\alpha_i})) = \begin{cases} \mathbb{Z} & , \text{ if } v_{\alpha_i} \text{ span a } p\text{-simplex} \\ 0 & , \text{ otherwise.} \end{cases}$$

Given $\sigma \in C^p(\underline{U}, \mathbb{Z})$, we can define a simplicial p-cochain σ' as follows:

For $\Delta = \langle v_{\alpha_0}, \cdots v_{\alpha_p} \rangle$ a p-simplex with vertices $v_{\alpha_0}, \cdots v_{\alpha_p}$,

$\sigma'(\Delta) = \sigma_{\alpha_0,\cdots\alpha_p}$.

$\Rightarrow$ $\sigma \longmapsto \sigma'$ gives an isomorphisms of Abelian groups

$C^p(\underline{U}, \mathbb{Z}) \xrightarrow{\phi} C^p(K, \mathbb{Z})$ → simplicial p-cochain group

$\phi(\sigma) = 0 \Rightarrow \sigma_{\alpha_0,\cdots\alpha_p} = 0. \Rightarrow \phi$ is mono.

Given a simplicial p-cochain τ,

$\tau(\Delta) \in \mathbb{Z}$. $\Delta = \langle v_{\alpha_0} \cdots v_{\alpha_p} \rangle$.

$\Rightarrow \sigma_{\alpha_0,\cdots\alpha_p} = \tau(\Delta). \Rightarrow \phi(\sigma) = \tau.$ $\quad \sigma = (\sigma_{\alpha_0,\cdots\alpha_p}) \in \prod \mathcal{F}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$

$\Rightarrow \phi$ is onto.

$$(\delta\sigma')(\langle \alpha_0, \cdots \alpha_{p+1} \rangle) = \sum_{i=0}^{p+1} (-1)^i \sigma'(\langle \alpha_0, \cdots \hat{\alpha}_i \cdots \alpha_{p+1} \rangle)$$

$$= \sum_{i=0}^{p+1} (-1)^i \sigma_{\alpha_0,\cdots\hat{\alpha}_i\cdots\alpha_{p+1}} = (\delta\sigma)_{\alpha_0,\alpha_1,\cdots\alpha_{p+1}} = (\delta\sigma)'(\langle \alpha_0, \alpha_1, \cdots \alpha_{p+1} \rangle)$$

so that we have an isomorphism of chain complexes

$C^*(\underline{U}, \mathbb{Z}) \longrightarrow C^*(K, \mathbb{Z})$.

$\Rightarrow$ Hence an isomorphism $H^*(\underline{U}, \mathbb{Z}) \longrightarrow H^*(K, \mathbb{Z})$.

Since we can ~~divided~~ subdivide the complex K to make the cover $\underline{U}$ of M arbitrarily fine without changing $H^*(K, \mathbb{Z})$, we finally obtain