For each $n \in \mathbb{Z}_+$, $K(\alpha_1, \alpha_2 \cdots \alpha_n)$ is an algebraic extension of K. $\Rightarrow$ By P261 (Hungerford) since every algebraic extension field of a field of characteristic 0 is seperable, and $K(\alpha_1 \cdots \alpha_n)$ is finite, by p287 Prop. 6.15.
$K(\alpha_1, \cdots \alpha_n)$ is a simple extension of K.

Thus $[K(\alpha_1 \cdots \alpha_n) : K] \leq d$, since $K(\alpha_1 \cdots \alpha_n) = K(\beta)$.

$\Rightarrow$ $[K(\alpha_1, \alpha_2 \cdots) : K] = \infty$ is impossible.

Comment: $[K(\alpha_1 \cdots \alpha_m) : K] \leq d$ even if $m > d$.

$\Rightarrow$ We can conclude that $[F : K] < \infty$.
$\Rightarrow$ Since F is seperable over K by p261, Remark, F is a simple extension of K by p287 ~ p288 (Primitive element theorem), i.e $F = K(r)$. $r \in F$.
$\Rightarrow$ $[F : K] = [K(r) : K] \leq d$ ⌟

- To complete the proof of our assertion, we want to exhibit a rational function on V which satisfies no polynomial relation of degree less than d over the field $\pi^* K(\mathbb{P}^k)$. To do this, we factor the projection map π: choose generic planes $\mathbb{P}^{n-k-2} \subset \mathbb{P}^{n-k-1}$ and $\mathbb{P}^k \subset \mathbb{P}^{k+1}$, and let $\pi' : V \longrightarrow \mathbb{P}^{k+1}$ be projection from $\mathbb{P}^{n-k-2}$.