$$[G, \bar\partial] = [G, \bar\partial^*] = 0,$$

and $\quad I = \mathcal{H} + \Delta G.$

3. Consequently, there is an isomorphism

$$\mathcal{H}^{p,q}(E) \longrightarrow H_{\bar\partial}^{p,q}(E), \qquad \text{and}$$

4. The $*$-operator gives an isomorphism

$$H^q(M, \Omega^p(E)) \cong H^{n-q}(M, \Omega^{n-p}(E^*))^*.$$

For $p=0$, this last result reads

$$H^q(M, \mathcal{O}(E)) \cong H^{n-q}(M, \mathcal{O}(E^* \otimes K_M))^*.$$

This isomorphism is called Kodaira–Serre duality.

Pf Since $\quad *_E \Delta = \Delta *_E,$

$$*_E : \mathcal{H}^{p,q}(E) \longrightarrow \mathcal{H}^{n-p,n-q}(E^*) \quad \text{is isomorphic.}$$

$$*_E(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = -*_E\bar\partial *_E\bar\partial *_E - *_E^2\bar\partial *_E\bar\partial$$

$$= \bar\partial^*\bar\partial *_E - \bar\partial *_E \bar\partial *_E^2 = \bar\partial^*\bar\partial *_E + \bar\partial\bar\partial^* *_E$$

$$(p,q) \longmapsto (p, q+1) \longrightarrow (n-p, n-q-1)$$

$$\longrightarrow (n-p, n-q) \quad \Rightarrow \quad *_E^2 = (-1)^{p+q} \quad *_E^2 = (-1)^{2n-p-q}$$

$$\Rightarrow \quad \mathcal{H}^{p,q}(E) \cong H_{\bar\partial}^{p,q}(E) \underset{\substack{\uparrow \\ \text{Dolbeault iso.}}}{\cong} H^q(M, \Omega^p(E))$$