Then, by the $\bar\partial$-Poincaré lemma the inclusion
$$\Omega^{p*} \longrightarrow \mathcal{A}^{p,*}$$
is a quasi-isomorphism.

⊩

$$\begin{tikzcd}
\Omega^p \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d, hook] & {} \\
\mathcal{A}^{p,0} \arrow[r] & \mathcal{A}^{p,1} \arrow[r] & \mathcal{A}^{p,2} \arrow[r] & \cdots
\end{tikzcd}$$

If $q>0$, since $\mathcal{A}^{p,q-1} \xrightarrow{\bar\partial} \mathcal{A}^{p,q} \xrightarrow{\bar\partial} \mathcal{A}^{p,q+1}$ is exact,
$$\mathcal{H}^q_x = \lim_{U \ni x} \frac{\ker\{\bar\partial : \mathcal{A}^{p,q}(U) \to \mathcal{A}^{p,q+1}(U)\}}{\bar\partial\ \mathcal{A}^{p,q-1}(U)} = 0$$

$\Rightarrow$ $\mathcal{H}^q(\mathcal{A}^{p,*}) = 0$

If $q=0$, $\mathcal{H}^0(\Omega^{p*}) = \Omega^p$
$\mathcal{H}^0(\mathcal{A}^{p,*}) = \bigcup \mathcal{H}^0_x$

where
$$\mathcal{H}^0_x = \lim_{U\ni x} \frac{\ker\{\bar\partial : \mathcal{A}^{p,0}(U) \to \mathcal{A}^{p,1}(U)\}}{0}$$
$$= \Omega^p_x .$$

Since j is inclusion, $\mathcal{H}^0(\Omega^{p*}) \overset{j_*}{\cong} \mathcal{H}^0(\mathcal{A}^{p,*})$ $j^* = id.$ ⌟

Repeating the argument just given for de Rham's theorem gives the Dolbeault isomorphism
$$H^q(M, \Omega^p) \cong H^{p,q}_{\bar\partial}(M).$$

⊩ By the lemma on P447, $\mathbb{H}^*(M, \Omega^{p*}) \cong \mathbb{H}^*(M, \mathcal{A}^{p,*})$ ③

$$\Rightarrow \quad ({}'E_{\Omega^{r}*})_2^{p,q} = H^p(M, \mathcal{H}^q(\Omega^{r}_*)) = \begin{cases} H^p(M, \Omega^r) & q=0 \\ 0 & q>0 \end{cases}$$