$$\mathcal{H}^q(\mathbb{C}^*) = \begin{cases} \mathbb{C}, & q=0 \\ 0, & q>0 \end{cases},$$

$$\mathcal{H}^q(\Omega^*) = \begin{cases} \mathbb{C} & q=0 \\ 0 & q>0 \end{cases}$$

since $\Omega(U) \xrightarrow{d} 0$

$h \longmapsto dh=0 \Rightarrow h$ is constant $\Rightarrow$

$$\ker d\,(U) \equiv \mathbb{C}.$$

$$\Rightarrow \quad j_* = id : \mathcal{H}^q(\mathbb{C}^*) \longrightarrow \mathcal{H}^q(\Omega^*).$$

Concerning the right-hand side of (*), the second spectral sequence has

$$''E_2^{p,q} = H_d^p(H^q(M, \Omega^p)).$$

$$H^q(M, \Omega^{p-1}) \xrightarrow{d} H^q(M, \Omega^p) \xrightarrow{d} H^q(M, \Omega^{p+1})$$

$$''E_2^{p,q} = \frac{\ker d}{\operatorname{Im} d}$$

Two cases are noteworthy: If M is compact Kähler, then $d=0$ on $H^q(M, \Omega^p) \cong H_{\bar\partial}^{p,q}(M)$, since $2\Delta_{\bar\partial} = \Delta_d$; thus $''E_2 = E_\infty$ and

$$\mathbb{H}^n(M, \Omega^*) \cong \bigoplus_{p+q=n} H^q(M, \Omega^p),$$

which is the Hodge decomposition.

Given any class $[a] \in H_{\bar\partial}^{p,q}(M)$, then it has a unique harmonic representative a. $\Rightarrow \Delta_{\bar\partial}\, a = 0 \Leftrightarrow \underset{0}{\underset{\|}{\bar\partial a}} = \bar\partial^* a$