First, in order for $i_L$ to be well-defined the linear system $|L|$ can not have any base points, i.e., for each $x \in M$ the restriction map

$$H^0(M, \mathcal{O}(L)) \xrightarrow{r_x} L_x$$

must be surjective.

⌈ $i_L : M \longrightarrow \mathbb{P}^N$

$x \longmapsto [s_0(x), \cdots s_N(x)]$

base-point free $\Rightarrow$ $\exists$ $s_i(x) \neq 0$ $\Rightarrow$ $r_x(s_i) = s_i(x) \neq 0$ in $L_x$. $\Rightarrow \{\alpha\, s_i(x)\} = L_x$. $\Rightarrow r_x$ is surjective. ⌋

Granted this, $i_L$ will be an embedding if

1. $i_L$ is one to one. Clearly this is the case $\Leftrightarrow$ for all $x$ and $y$ in $M$, there exists a section $s \in H^0(M, \mathcal{O}(L))$ vanishing at $x$ but not at $y$, i.e. $\Leftrightarrow$ the restriction map

(*) $$H^0(M, \mathcal{O}(L)) \xrightarrow{r_{x,y}} L_x \otimes L_y$$

is surjective for all $x \neq y \in M$.

⌈ (i) ($\Leftarrow$). Clear ($\Rightarrow$). $i_L(x) \neq i_L(y) \Leftrightarrow [s_0(x), \cdots s_N(x)] \neq [s_0(y), \cdots s_N(y)]$ $\Rightarrow [s_0(x), s_1(x)] \neq [s_0(y), s_1(y)]$ say $0.1$. Suppose $s_0(x) = 0$. $s_0(y)$ must be nonzero. If $s_1(x) = 0$, $s_1(y) \neq 0$. $s_0(x) \neq 0 \neq s_1(x)$.

$\Rightarrow \exists\ \alpha$ s.t $s_0(x) + \alpha s_1(x) = 0$. $\Rightarrow$ $s_0(y) + \alpha s_1(y) \neq 0$.

$\Rightarrow$ $s_0 + \alpha s_1$ is a section satisfying the above condition.

(ii). $s_1(x) = 0$, $s_1(y) \neq 0$. & $s_2(x) \neq 0$, $s_2(y) = 0$.

Consider $s = s_1 + s_2$ $\Rightarrow$ $s(x) = s_2(x) \neq 0$. $s(y) = s_1(y) \neq 0$.