$$\begin{tikzcd} \tilde{L}^K = \pi^* L^K \arrow[r] \arrow[d] & L^K \arrow[d] \\ \tilde{M} \arrow[r, "\pi"] & M \end{tikzcd}$$

$s \in H^0(M, \mathcal{O}_M(L^K))$

$\pi^* s \in H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^K))$.

For any global section $\tilde{\sigma}$ of $\tilde{L}^K$, the section of L^K given by σ over $M - \{x, y\}$ extends by Hartogs theorem to a global section $\sigma \in H^0(M, \mathcal{O}(L^K))$, and so we see that π^* is an isomorphism.

For a global section $\tilde{\sigma}$ of $\tilde{L}^K$, consider the restriction to $\tilde{M} - E$.

$$\begin{tikzcd} \tilde{L}^K|_{\tilde{M}-E} \arrow[r, "\cong"] \arrow[d] & L^K|_{M-\{x,y\}} \arrow[d, shift right] \\ \tilde{M}-E \arrow[r, "\pi", "\cong"'] \arrow[u, bend left, "\tilde{\sigma}|"] & M-\{x,y\} \arrow[u, shift right, "\sigma"'] \end{tikzcd}$$

Let σ be the section corresponding to $\tilde{\sigma}|$.

$\Rightarrow$ σ can be extended to M by Hartogs' theorem by considering the trivializations of the nbds of x & y. Furthermore, σ is uniquely extended by continuity. $\Rightarrow$ π^* is an isomorphism.

Furthemore, by definition $\tilde{L}^K$ is trivial along E_x and E_y, i.e.,

$$(\tilde{L}^K)|_{E_x} = E_x \times L_x^K, \quad (\tilde{L}^K)|_{E_y} = E_y \times L_y^K,$$

so that $H^0(E, \mathcal{O}_E(\tilde{L}^K)) \cong L_x^K \oplus L_y^K$, and if r_E denotes the restriction map to E, the diagram

$$\begin{tikzcd} H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L}^K)) \arrow[r, "r_E"] & H^0(E, \mathcal{O}_E(\tilde{L}^K)) \arrow[d, equal] \\ H^0(M, \mathcal{O}(L^K)) \arrow[u, "\pi^*", "\cong"'] \arrow[r, "r_{x,y}"] & L_x^K \oplus L_y^K \end{tikzcd}$$

commutes.