The metric connections of hermitian vector bundles behave well w.r.t bundle operations, as we see in the next two lemmas.

lemma: $E \longrightarrow M$ hermitian v.b. $F \subset E$ holomorphic subbundle $\Rightarrow$ F itself a hermitian bundle with metric connection D_F. On the other hand, the metric connection D_E in E and direct-sum decomposition $E = F \oplus F^{\perp}$ induced by the metric give a connection $\pi_F D_E$ in F, and

$$D_F = \pi_F \circ D_E. \quad \text{where } \pi_F \text{ is the projection onto } F.$$

pf) If ξ is a section of F, then $(\pi_F \circ D_E)''(\xi) = \pi_F(D_E''\xi)$ $= \pi_F(\bar\partial \xi) = \bar\partial \xi$, so that $\pi_F \circ D_E$ is compatible with the complex structure.

$$\begin{array}{ccccc}
\mathcal{A}^0(E) & \xrightarrow{D_E} & \mathcal{A}^1(E) = \mathcal{A}^1(F \oplus F^{\perp}) & \xrightarrow{\pi_F} & \mathcal{A}^1(F) \\
\| & & \| & & \| \\
\mathcal{A}^0(F) \oplus \mathcal{A}^0(F^{\perp}) & & \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E) & & \mathcal{A}^{1,0}(F) \oplus \mathcal{A}^{0,1}(F) \\
\cup & & \downarrow \qquad\qquad \downarrow & & \\
\xi & & \mathcal{A}^{1,0}(F) \qquad \mathcal{A}^{0,1}(F) & &
\end{array}$$

Γ $\pi_F \circ D_E(\xi) = (\pi_F \circ D_E)'(\xi) \oplus (\pi_F \circ D_E)''(\xi)$
$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$
$\pi_F \circ (D_E'\xi \oplus D_E''\xi) = \pi_F \circ D_E'(\xi) \oplus \pi_F D_E''(\xi)$

$\Rightarrow \quad (\pi_F \circ D_E)''(\xi) = \pi_F \circ D_E''(\xi). = \pi_F(\bar\partial \xi)$

Since $\bar\partial \xi \subset \mathcal{A}^{0,1}(F)$, $\quad \pi_F(\bar\partial\xi) = \bar\partial \xi$. $\lrcorner$)

$\xi, \xi' \in \mathcal{A}^0(F)$, $\quad d\langle \xi, \xi' \rangle = \langle D_E\xi, \xi' \rangle + \langle \xi, D_E\xi' \rangle$
$= \langle \pi_F \circ D_E\xi, \xi' \rangle + \langle \xi, \pi_F \circ D_E \xi' \rangle$ since $\xi' \in \mathcal{A}^0(F)$ and

$\langle \pi_{F^{\perp}} \circ D_E\xi, \xi' \rangle = 0$, so that $\pi_F \circ D_E$ is compatible with the metric.