Local duality. We now come to the local duality theorem. Given $U$ a sufficiently small nbd of the origin and $f: U \to \mathbb{C}^n$ with $f^{-1}(0) = \{0\}$, or equivalently given an ideal $I = I(f) = \{f_1, \cdots f_n\}$ in the local ring $\mathcal{O} = \mathcal{O}_{\{0\}}$ at the origin and having $\{0\}$ as isolated common zero of the $f_i$'s, we may use the property

$$\operatorname{Res}_{\{0\}} \left( \frac{g\, dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n} \right) = 0 \qquad \text{for } g \in I,$$

to define a symmetric pairing

$$\operatorname{res}_f : \mathcal{O}/I \otimes \mathcal{O}/I \longrightarrow \mathbb{C}$$

by setting

$$\operatorname{res}_f(g, h) = \operatorname{Res}_{\{0\}} \left( \left( \frac{1}{2\pi\sqrt{-1}} \right)^n \frac{g(z)\, h(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} \right).$$

The basic result is the

Local Duality Theorem I. The pairing "$\operatorname{res}_f$" is nondegenerate; i.e., if

$$\left( \frac{1}{2\pi\sqrt{-1}} \right)^n \int_{|f_i(z)| = \varepsilon} \frac{g(z)\, h(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} = 0$$

for all $h(z) \in \mathcal{O}$, then $g(z)$ lies in the ideal $\{f_1, \cdots f_n\}$.

Proof. The proof is based on the transformation law and two further general results in local analytic geometry. The first of these is that $f_1, \cdots f_n \in \mathcal{O}$ form a regular sequence, which by definition means that

$f_i$ is not a zero divisor in $\mathcal{O}/\{f_1, \cdots f_{i-1}\}$ $(1 \le i \le n)$.