$P_1, P_2, \cdots P_8$ do not give independent conditions on cubics.

⑤ $L_{12}$ doesn't contain any point.

$$L_{12} + L_{45} + P_{67} \ni P_3$$

$\Rightarrow$ (i) $P_3 \in L_{45}$ $\Rightarrow$ By the case ③, done.

(ii) $P_3 \in P_6$ $\Rightarrow$ By the case ①, done.

③ $P_1$ is infinitely near $P_2$, which is itself infinitely near $P_3$.

As we expect in the arguments above, we have to show that $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ impose linearly independent conditions on cubics unless five are collinear.

$$\frac{\partial \tau}{\partial x} + \frac{\partial \tau}{\partial y} g'(x) = 0$$

$\Rightarrow$ I think, this is not a correct interpretation for $P_1$. See P752 note.

$$\Rightarrow \left(\frac{\partial^2 \tau}{\partial x^2}, \frac{\partial^2 \tau}{\partial x \partial y}\right) \cdot (1, g'(x)) + \left(\frac{\partial^2 \tau}{\partial x \partial y}, \frac{\partial^2 \tau}{\partial y^2}\right) \cdot (1, g'(x)) g'(x)$$

$$+ \frac{\partial \tau}{\partial y} g''(x) = 0,$$ gives a condition on cubics.

We know that $P_2, P_3, P_4, P_5 \cdot P_7$ impose linearly independent conditions, unless five are collinear. If $P_1, P_2, P_3, \cdots P_7$ fail to impose linearly independent condition, then any cubic curve containing $P_2, P_3, \cdots P_7$ contains $P_1$.

$P_1$ should be interpreted as the second order tangent direction at $P_3$. Since $P_1 \in \mathbb{P}^1$ at $P_3$, any cubic has nonzero second derivative. $\Rightarrow$