it follows that $\mathrm{Tor}^{\mathcal{O}}_{n+k}(\mathbb{C}, M) = 0$ for $k \geq 1$. Q.E.D.

⌈ $\mathrm{Tor}^{\mathcal{O}}_q(\mathbb{C}, M) \equiv H_q(K.\otimes_{\mathcal{O}} M) = 0$ if $q \geq n+1$

$\Rightarrow \mathrm{Tor}^{\mathcal{O}}_{n+1}(\mathbb{C}, M) = H_{n+1}(K.\otimes_{\mathcal{O}} M) = 0 = \mathrm{Tor}^{\mathcal{O}}_1(\mathbb{C}, F) = 0$

$\Rightarrow$ By the lemma, F is free $\mathcal{O}$-module. ⌋

A Brief Tour Through Coherent Sheaves.

Definitions and Elementary Properties. On an open set $U \subset \mathbb{C}^n$ we now denote by $\mathcal{O}$ the sheaf of holomorphic functions and by $\mathcal{O}_z = \varinjlim_{z \in V} \mathcal{O}(V)$ the stalk of $\mathcal{O}$ at a point $z \in U$. A sheaf mapping $\mathcal{O}^{(p)} \xrightarrow{F} \mathcal{O}^{(q)}$ is given by a $(p\times q)$ matrix of holomorphic functions defined on U. We define the sheaf of $\mathcal{O}$-modules $\mathcal{R}$ by

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}^{(p)} \xrightarrow{F} \mathcal{O}^{(q)}.$$

In the discussion of local rings, we pointed out that, because of the Noetherian property of $\mathcal{O}_{z_0}$, $\mathcal{R}_{z_0}$ was a finitely generated $\mathcal{O}_{z_0}$-module. The following fundamental lemma is due to Oka:

Oka's Lemma. The sheaf $\mathcal{R}$ is locally finitely generated as a sheaf of $\mathcal{O}$-modules. More precisely, if $r_1, \cdots r_m$ are sections of $\mathcal{R}$ in a nbd of z_0 that generate the $\mathcal{O}_{z_0}$-module $\mathcal{R}_{z_0}$, then they generate the $\mathcal{O}_z$-module $\mathcal{R}_z$ for $\|z - z_0\| < \varepsilon$.