We will now use smoothing to prove some regularity results concerning the Laplace equation on distributions

$$\Delta T = S,$$

where
$$\Delta = -\sum_i \frac{\partial^2}{\partial x_i^2}$$

Lemma. If $T \in \mathcal{D}(\mathbb{R}^n)$ satisfies $\Delta T = 0$, then $T = T_\varphi$ for some $\varphi \in C^\infty(\mathbb{R}^n)$ with $\Delta\varphi = 0$.

Proof. Smooth functions φ satisfying $\Delta\varphi = 0$ are said to be harmonic. We shall first prove that harmonic functions obey the mean-value property

$$\varphi(y) = \int_{\|x-y\|=\epsilon} \varphi(x)\, \sigma_y(x)$$

where, if
$$\sigma = C_n \frac{*(r\,dr)}{r^n}$$

is the form encountered in the preceding section, then

$$\sigma_y(x) = \sigma(x-y)$$
is the invariant volume form on the sphere $\|x-y\| = \epsilon$ having total area 1.

Here the invariant form means that $R_g^* \sigma = \sigma$, $g \in \{A : A$ 2×2 orthogonal matrix with det $1\}$, $\sigma \in A^1(\mathbb{R}^2 - \{0\})$.
In general, $R_g^*\sigma = \sigma$, g is $n\times n$ orthogonal matrix with det 1, i.e., g is a proper orthogonal matrix.
Let's prove it for $n = 2$.

$$g = \begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix} \qquad \sigma = \frac{x\,dy - y\,dx}{x^2+y^2}.$$