lines in $\mathbb{P}^2$ tangent to two conics.

「 By changing coordinates on $\mathbb{P}^2$, we may assume that $p_1, p_2, p_3$ are $[1,0,0]$, $[0,1,0]$, $[0,0,1]$ respectively, and the quadratic transformation $\phi$ is

$$\mathbb{P}^2 \longrightarrow \mathbb{P}^2$$
$$[X_0, X_1, X_2] \longmapsto [X_0X_1,\ X_0X_2,\ X_1X_2].$$

see P496 ~ P497.

$$a_{00}X_0^2 + a_{11}X_1^2 + a_{22}X_2^2 + a_{01}X_0X_1 + a_{02}X_0X_2 + a_{12}X_1X_2 = 0$$

passes $p_1, p_2, p_3 \Rightarrow a_{00} = a_{11} = a_{22} = 0$.

$\Rightarrow$ Let $Y_0 = X_0X_1$, $Y_1 = X_0X_2$ and $Y_2 = X_1X_2$.

$\Rightarrow \quad a_{01}Y_0 + a_{02}Y_1 + a_{12}Y_2 = 0$

$\Rightarrow \quad \phi(\text{conic}) = \text{line}.$

Given a line $aX_0 + bX_1 + cX_2 = 0$,

$$a\frac{Y_0}{X_1} + bX_1 + cX_2 = 0.$$

$\Rightarrow \quad aY_0 + bX_1^2 + cY_2 = 0 \Rightarrow aY_0 + b\dfrac{Y_0Y_2}{Y_1} + cY_2 = 0$

$\Rightarrow \quad aY_0Y_1 + bY_0Y_2 + cY_1Y_2 = 0$

$\Rightarrow \quad \phi(\text{line}) = \text{conic passing } p_1, p_2, p_3.$

$$\begin{array}{ccc}
\mathbb{P}^2 & \longrightarrow & \mathbb{P}^2 \\
\text{through } p_1,p_2,p_3 \ \cup & & \cup \ \text{any} \\
(\text{conic})\ C & \longrightarrow & l\ (\text{line}) \\
\text{tangent to} & & \text{tangent to} \\
l_1, l_2 & \longrightarrow & C_1, C_2
\end{array}$$

Thus the # of conics through $p_1, p_2, p_3$ tangent to $l_1, l_2$