$$\begin{aligned}\tau' &= d\varphi' - \theta'^{*}\wedge\varphi' \\ &= d(g^{*}\varphi) - \theta'^{*}\wedge g^{*}\varphi \\ &= d(g^{*})\cdot\varphi + g^{*}\varphi - g^{*}\cdot\theta^{*}\cdot {}^{t}g\cdot g^{*}\cdot\varphi - d(g^{*})\cdot {}^{t}g\cdot g^{*}\cdot\varphi \\ &= g^{*}\varphi - g^{*}\cdot\theta^{*}\wedge\varphi \\ &= g^{*}\tau\end{aligned}$$

i.e., the quantity

$$\tilde{\tau} = \sum \tau_i \otimes v_i \in A^{2,0}(T')$$

is a tensor invariant of the metric, called the torsion tensor.

$\ulcorner$ Let $\varphi_i' = \sum g_{ij}^{*}\varphi_j$.

$$\begin{aligned}\delta_{il} = \varphi_i'(v_l') &= \sum g_{ij}^{*}\varphi_j(v_l') = g_{ij}^{*}\varphi_j(g_{lk}v_k) = g_{ij}^{*}g_{lk}\delta_{jk} \\ &= g_{ij}^{*}g_{lk}\delta_{jk} = g_{ij}^{*}g_{lj} = g_{ij}^{*}({}^{t}g)_{jl} = (g^{*}\,{}^{t}g)_{il}\end{aligned}$$

$$\Rightarrow \quad g^{*}\,{}^{t}g = I \quad \Rightarrow \quad g^{*} = {}^{t}g^{-1}.$$

$$\begin{aligned}D^{*}\varphi_i' &= \theta_{ij}'^{*}\varphi_j' = D^{*}(g_{il}^{*}\varphi_l) = dg_{il}^{*}\otimes\varphi_l + g_{il}^{*}D^{*}\varphi_l \\ &= dg_{il}^{*}\otimes\varphi_l + g_{il}^{*}\theta_{lk}^{*}\otimes\varphi_k \\ &= dg_{il}^{*}\otimes\underline{\varphi_l} + g_{ik}^{*}\theta_{kl}^{*}\otimes\varphi_l \\ &= dg_{il}^{*}\otimes g_{lm}^{*-1}\varphi_m' + g_{ik}^{*}\theta_{kl}^{*}g_{lm}^{*-1}\varphi_m' \\ &= (dg^{*}g^{*-1})_{ij}\varphi_j' + (g^{*}\theta^{*}g^{*-1})_{ij}\varphi_j'\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad \theta'^{*} &= g^{*}\theta^{*}g^{*-1} + (dg^{*})g^{*-1} \qquad \text{by } \underline{(g^{*-1} = {}^{t}g)} \\ &= g^{*}\theta^{*}\,{}^{t}g + (dg^{*})\,{}^{t}g.\end{aligned}$$

$$\begin{aligned}\tau' &= d\varphi' - \theta'^{*}\wedge\varphi' = d(g^{*}\varphi) - \theta'^{*}\wedge g^{*}\varphi = (dg^{*})\varphi + g^{*}d\varphi \\ &\quad - (g^{*}\theta^{*}\,{}^{t}g + dg^{*}({}^{t}g))\wedge g^{*}\varphi \\ &= \cancel{(dg^{*})\varphi} + g^{*}d\varphi - g^{*}\theta^{*}\varphi - \cancel{dg^{*}\varphi} \\ &= g^{*}d\varphi - g^{*}\theta^{*}\varphi = g^{*}(d\varphi - \theta^{*}\wedge\varphi) = g^{*}\cdot\tau\end{aligned}$$