We will not prove the bilinear relations in full generality, but will verify them in some cases including all those to be used in our applications to geometry. (The general proof may be based on the following observations:

In the full exterior algebra

$$V = \Lambda^* \mathbb{C}^n \otimes \Lambda^* \overline{\mathbb{C}^n}$$

corresponding to the differential forms at a point $x \in M$, there is an sl_2-action given by $\{L, \Lambda, h\}$ as above. Decomposing V under the action of the unitary group U_n and thus by Schur's lemma any U_n-invariant quadratic form on $P^k V$ is necessarily definite. The primitive harmonic forms on M are those which lie in $P^k V$ (fixed k) for each $x \in M$, and this yields a proof. The result that decomposing V under sl_2 together with $\Pi^{(p,q)}$ yields the same irreducible factors as under the action of U_n is proved in Herman Weyl's book The Classical Groups – it implies that, in general, there are no further Hodge identities.)

First, let M be a compact Riemann surface. By the Hodge decomposition,

$$\begin{aligned} H^1(M, \mathbb{C}) &= H^{1,0}(M) \oplus H^{0,1}(M) \\ &\cong H^0(M, \Omega^1) \oplus \overline{H^0(M, \Omega^1)}. \end{aligned}$$

The number of independent holomorphic 1-forms on M (classically called differentials of the first kind) is thus equal to $b_1(M)/2$; this in fact was one of the first links established between the topology of a complex manifold and its analytic structure. To verify the bilinear relations for M let $\xi = h(z)\, dz \in H^{1,0}(M)$;