For a general $T \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}(DT)_\epsilon(\varphi) &= (DT)(\varphi_\epsilon)\\ &= T(-D\varphi_\epsilon)\\ &= T(-D\varphi)_\epsilon \qquad \text{(by the previous step)}\\ &= T_\epsilon(-D\varphi)\\ &= DT_\epsilon(\varphi),\end{aligned}$$

which proves assertion 3.

「 $(DT)_\epsilon(\varphi) = (DT)(\varphi_\epsilon)$ (by the assertion 2)
$= T(-D\varphi_\epsilon)$ (by P 367 Exam. 2)
$= T(-D\varphi)_\epsilon$. ((since $(-D\varphi_\epsilon)(x) = -D\int \varphi(y)$

$\chi_\epsilon(x-y)\,dy = -\int \varphi(y) \frac{\partial}{\partial x_i}\chi_\epsilon(x-y)\,dy$

$= -\int \varphi(x-u) \frac{\partial}{\partial u_i}\chi_\epsilon(u)\,du \cdot (-1)^n$ (let $x-y=u$)

$= +\int \frac{\partial \varphi(x-u)}{\partial u_i}\chi_\epsilon(u)\,du\,(-1)^n$ (by integration by parts)

$= -\int \frac{\partial \varphi(y)}{\partial y_i}\chi_\epsilon(x-y)\,dy\,(-1)^n(-1)^n$ (let $x-u=y$)

$= -\int D\varphi(y)\,\chi_\epsilon(x-y)\,dy = -(D\varphi)_\epsilon(x)$))

$\Rightarrow$ $T(-D\varphi)_\epsilon = T(-(D\varphi)_\epsilon) = T_\epsilon(-D\varphi)$ (by the assertion 2)

$= (DT_\epsilon)(\varphi)$.

$\Rightarrow$ We proved $(DT)_\epsilon = D(T_\epsilon)$ for $D = \partial/\partial x_i$.

$\left(\frac{\partial^2}{\partial x_1 \partial x_2} T\right)_\epsilon = \frac{\partial}{\partial x_1}\left(\left(\frac{\partial}{\partial x_2}T\right)_\epsilon\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial}{\partial x_2}(T_\epsilon)\right)$

$= \frac{\partial^2}{\partial x_1 \partial x_2}(T_\epsilon)$, In this way, we can prove the general case. 」