$\Rightarrow$ adjoint of ∂ on dz_I's, dz_J's.

It plays the same role as ∂''^* on $dz_I \wedge d\bar{z}_J$.

$\Rightarrow$ According to P 113, $\quad \partial''^* = -\sum \bar{\partial}_k i_k = \partial'^*$

$\Rightarrow$ Thus $\quad -i\,\partial'^* = -i\,\partial^*$ at z_0. $\quad$ ⅃

Now, some consequences: if $\Delta_d = dd^* + d^*d$ is the d-Laplacian, we have

$$[L, \Delta_d] = 0 \quad \text{or equivalently}$$
$$[\Lambda, \Delta_d] = 0.$$

pf) First note that since ω is closed,

$$-d(L\eta) = d(\omega \wedge \eta) = \omega \wedge d\eta = -L(d\eta)$$

i.e $\quad [L, d] = 0$

and so by taking adjoints $\quad [\Lambda, d^*] = 0$.

Now $\quad \Lambda(dd^* + d^*d) = \Lambda dd^* + \Lambda d^*d$, since $[\Lambda, d] = -4\pi d^{c*}$

$$\begin{aligned}
&= (d\Lambda - 4\pi d^{c*})d^* + \Lambda d^*d = d\Lambda d^* - 4\pi d^{c*}d^* + \Lambda d^*d \\
&= d\Lambda d^* + 4\pi d^* d^{c*} + \Lambda d^*d \\
&= d\Lambda d^* + 4\pi d^* d^{c*} + d^*\Lambda d \\
&= dd^*\Lambda + d^*(4\pi d^{c*} + \Lambda d) = dd^*\Lambda + d^*d\Lambda = \Delta\Lambda. \quad \text{Q.E.D.}
\end{aligned}$$

We also have, as mentioned earlier,

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial}.$$

pf) First we show that $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$: since $\Lambda\partial - \partial\Lambda = i\bar{\partial}^*$, we have

$$\begin{aligned}
i(\partial\bar{\partial}^* + \bar{\partial}^*\partial) &= \partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\partial \\
&= \partial\Lambda\partial - \partial^2\Lambda - \partial\Lambda\partial = 0
\end{aligned}$$