is the isomorphism

$$\mathcal{H}^{p,q}(M) \cong H^{p,q}_{\bar\partial}(M);$$

if this can be proved, then we will have a unique representative for each cohomology class, which should certainly be an advantage. The isomorphism (*) is part of the Hodge Theorem, whose proof together with the corollaries of (*) will ~~occur~~ occupy this section.

We begin by giving an explicit formula for the adjoint $\bar\partial^*$, thereby proving its existence. First we define the star, or duality operator,

$$* : A^{p,q}(M) \longrightarrow A^{n-p,\,n-q}(M).$$

by requiring

$$(\psi(z), \eta(z))\, \Phi(z) = \psi(z) \wedge * \eta(z) \quad \text{for all } \psi \in A^{p,q}(M).$$

This is an algebraic operator, which is given locally as follows: if we write

$$\eta = \sum_{I,J} \eta_{I\bar J}\, \varphi_I \wedge \bar\varphi_J$$

[we have only to check one term $\eta_{I\bar J}\, \varphi_I \wedge \bar\varphi_J$, where $\#I = p$, $\#J = q$.

$*\eta = ?$ let $\psi(z) = \sum \psi_{I\bar J}\, \varphi_I \wedge \bar\varphi_J$ $\quad I = \{i_1 \cdots i_p\}$, $i_1 < \cdots < i_p$

$$\langle \psi(z), \eta_{I\bar J}\, \varphi_I \wedge \bar\varphi_J \rangle\, \Phi(z) = \psi(z) \wedge * \eta(z) \quad \text{-- } (*)$$

let $* \eta(z) = \sum b_{I\bar J}\, \varphi_I \wedge \bar\varphi_J$.

left-hand side of (*) $= \psi_{I\bar J}\, \bar\eta_{I\bar J}\, \cancel{2^{p+q}}\, \Phi(z)$

$\;\;\;\|$

RHS of (*).

$\Rightarrow$ $* \eta(z) = b_{I^\circ \bar J^\circ}\, \varphi_{I^\circ} \wedge \bar\varphi_{J^\circ}$ is the only term remaining

$$\psi_{I\bar J}\, b_{I^\circ \bar J^\circ}\, \varphi_I \wedge \bar\varphi_J \wedge \varphi_{I^\circ} \wedge \bar\varphi_{J^\circ} = \psi_{I\bar J}\, \bar\eta_{I\bar J}\, \cancel{2^{p+q}}\, \Phi(z).$$