Let $U' \subset U$ be the locus of $|(\partial f_i/\partial z_j)_{1\le i,j\le k}| \neq 0$ and V' the locus $f_1 = \cdots = f_k = 0$. Then $V' = V \cap U'$ is a complex submanifold of U', and for any holomorphic function f vanishing on V the differential $df \equiv 0$ on V', i.e. f is constant on V'.

⌈ $f \equiv 0$ on V. $V' = V \cap U'$ submanifold of U'
$\Rightarrow$ $f \equiv 0$ on V' $\Rightarrow$ $df \equiv 0$ on V' ⌋

It follows that for $q \in V'$ near p, $V = V'$ is a manifold in a nbd of q and so $V_s \subset (|(\partial f_i/\partial z_j)_{1\le i,j\le k}| = 0)$.

Q.E.D.

~~Let $V = \{f_1 = f_2 = \cdots = f_\ell = 0\}$ on an open set $U \subset \mathbb{C}^n$. For $p \in V$, let k be the largest integer such that there exist k f_i's on U~~

Proof. For $p \in V$, let $V = \{f_1 = f_2 = \cdots = f_\ell = 0\}$ on an open set $U^{\ni p} \subset \mathbb{C}^n$. Let k be the largest integer such that there exist k f_i's on U and s.t. $J(f)$ has a $k \times k$ minor not everywhere singular on V. We may assume that $\left|\left(\frac{\partial f_i}{\partial z_j}\right)_{1\le i,j\le k}\right| \not\equiv 0$ on V.