Is it well-defined ? Yes, since $g_{\alpha\beta} S_{i,\beta} = S_{i,\alpha}$.

Since each $S_{i,\alpha}$ is holomorphic, f is holomorphic.

Locally, D_{λ_1,λ_2} is given as below.

$$\left\{ \frac{S_{2,\alpha}}{S_{1,\alpha}}(z_1 \cdots z_n) = -\lambda_1 \right\} \cap \left\{ \frac{S_{3,\alpha}}{S_{1,\alpha}}(z_1 \cdots z_n) = -\lambda_2 \right\}.$$

Singular point P_{λ_1,λ_2} in D_{λ_1,λ_2} means that

$$\frac{\partial \frac{S_{2,\alpha}}{S_{1,\alpha}}}{\partial z_i}(P_{\lambda_1,\lambda_2}) = 0 \quad \text{for all } i \quad \text{or}$$

$$\frac{\partial \frac{S_{3,\alpha}}{S_{1,\alpha}}}{\partial z_i}(P_{\lambda_1,\lambda_2}) = 0 \quad \text{for all } i.$$

Assume $\lambda_1 \neq 0, \infty$ $\lambda_2 \neq 0, \infty$. ($\because \#(D_{\lambda_1,\lambda_2}\text{'s}) < \infty$ in case $\lambda_i = 0, \infty$)

$$\left\{ \frac{\partial \frac{S_{2,\alpha}}{S_{1,\alpha}}}{\partial z_i}(z_1 \cdots z_n) = 0 \right\} = A_{\lambda_1,\lambda_2}$$

$$\left\{ \frac{\partial \frac{S_{3,\alpha}}{S_{1,\alpha}}}{\partial z_i}(z_1 \cdots z_n) = 0 \right\} = B_{\lambda_1,\lambda_2}$$

We have only to show that $\bigcup A_{\lambda_1,\lambda_2}$ $\bigcup B_{\lambda_1,\lambda_2}$ meets only a finite # of D_{λ_1,λ_2}'s .
To do this, we need to prove that each $\bigcup A_{\lambda_1,\lambda_2}$ meets only a finite # of D_{λ_1,λ_2}'s.
Again, we can prove that the singular locus V minus B is analytic subvariety of M.
$\Rightarrow$ $V - B = \bigcup C_\alpha$. $\Rightarrow$ Again by the same argument, we can conclude $\bigcup A_{\lambda_1,\lambda_2}$ meets only a finite # of D_{λ_1,λ_2}'s
$\Rightarrow$ We proved the desired.