$K \ni h. \Rightarrow$ Since GL_n is connected, $GL_n = K$.

Given any matrix $g \in M_n$, we can find $h \in GL_n$ s.t $I + h - g \in GL_n$, since GL_n is open and dense in M_n.

$$\Rightarrow 0 = \sum_i P(A_1, \dots, (h-g)A_i - A_i(h-g), \cdots A_k)$$
$$= \sum_i P(A_1, \dots, hA_i - A_i h - gA_i + A_i g, \cdots A_k)$$
$$= \sum_i P(A_1, \cdots, hA_i - A_i h, \cdots A_k) + \sum_i P(A_1, \cdots, A_i g - gA_i, \cdots A_k)$$
$$\Rightarrow \sum_i P(A_1, \cdots, A_i g - gA_i, \cdots A_k) = 0 \quad \cdots (**)$$

Thus $(**)$ is valid for any matrix $g \in M_n$." ⌟

Now we can prove Part 2. Let $D, \tilde{D}$ be two connections on E, with local connection and curvature matrices θ_α and $\tilde{\theta}_\alpha$, Θ_α and $\tilde{\Theta}_\alpha$. In terms of the trivialization φ_α, we have

$$D\xi_\alpha = d\xi_\alpha + {}^t\theta_\alpha \xi_\alpha, \qquad \tilde{D}\xi_\alpha = d\xi_\alpha + {}^t\tilde{\theta}_\alpha \xi_\alpha;$$

Consequently the operator $\eta = D - \tilde{D}$ is linear over $C^\infty(M)$, and so it is given in terms of the trivialization φ_α as multiplication by the transpose of the matrix $\eta_\alpha = \theta_\alpha - \tilde{\theta}_\alpha$, which transform by the rule

$$\eta_\alpha = g_{\alpha\beta}\, \eta_\beta\, g_{\alpha\beta}^{-1}.$$

where $g_{\alpha\beta} = \varphi_\alpha \cdot \varphi_\beta^{-1}$. Consider the homotopy

$$D_t = \tilde{D} + t\eta, \qquad 0 \le t \le 1,$$