Suppose $\sigma_\alpha(p)=0$ and $S_{0\alpha}(p)=0$. $\Rightarrow$ $\mathrm{ord}_{S,p}\left(\frac{\sigma_\alpha}{S_{0\alpha}}\right)\geq 0$

$\Rightarrow$ Since $\mathrm{ord}_{S,p}$ is independent of p.

$\mathrm{ord}_S\left(\frac{\sigma_\alpha}{S_{0\alpha}}\right)\geq 0$. $\Rightarrow$ $\left(\frac{\sigma_\alpha}{S_{0,\alpha}}\right)$ is holomorphic section of

$\Lambda^2 T^{*\prime}\mathbb{P}^2 = K_{\mathbb{P}^2}$ $\Rightarrow$ This is impossible unless $\left(\frac{\sigma_\alpha}{S_{0,\alpha}}\right)$ is zero section, since $H^0(\mathbb{P}^2, \Omega^2)=0$.

$\Rightarrow$ $\sigma=0$

Thus we can conclude that $\exists$ no point $p\in S$ s.t $\sigma_\alpha(p)=0=S_{0,\alpha}(p)$, which implies that $\left(\frac{\sigma_\alpha}{S_{0\alpha}}\right)$ has a simple pole along S, and holomorphic elsewhere.

According to P147, we get

$$(-1)^{i+1}\, g(z_1,z_2)\,\frac{dz_1\wedge\cdots\wedge \widehat{dz_i}\wedge dz_2}{\frac{\partial f}{\partial z_i}}\Bigg|_{f=0}$$

$\Rightarrow$ $i=0+1$, $\quad g(z_1,z_2)\dfrac{dz_2}{\frac{\partial f}{\partial z_1}}$ $\qquad i=2$, $-g(z_1,z_2)\dfrac{dz_1}{\frac{\partial f}{\partial z_2}}$ 」

Recall also that the Poincare residue map gives in this case an isomorphism

$$H^0(\mathbb{P}^2, \Omega^2(S)) \longrightarrow H^0(S, \Omega^1_S).$$

「 By P147~P148, we have the exact sequence

$$H^0(\mathbb{P}^2,\Omega^2)\longrightarrow H^0(\mathbb{P}^2, \Omega^2(S)) \xrightarrow{P.R} H^0(S, \Omega^1_S) \xrightarrow{\delta} H^1(\mathbb{P}^2, \Omega^2)$$

$\|$ $\qquad\qquad$ $\|$

0 $\longleftarrow$ see P49 or P118 $\longrightarrow$ 0

$\Rightarrow$ $H^0(\mathbb{P}^2, \Omega^2(S)) \longrightarrow H^0(S, \Omega^1_S)$ is an isomorphism. 」

Now consider $\omega\in H^0(\mathbb{P}^2, \Omega^2(S))$, written as above.