Proposition. Any curve $D \in |K+L+L'|$ that passes through all but one point of $C \cdot C'$ necessarily contains that remaining point.

Proof. If $\sigma \in H^0(S, \mathcal{O}(L))$ and $\sigma' \in H^0(S, \mathcal{O}(L'))$ define C and C', and if $\psi \in H^0(S, \mathcal{O}(K+L+L')) = H^0(S, \Omega^2(L+L'))$, then

$$\omega = \frac{\psi}{\sigma \cdot \sigma'}$$

is a meromorphic 2-form on S with polar curve $C+C'$ to which the general residue theorem

$$\sum_{p_\nu \in C \cap C'} \operatorname{Res}_{p_\nu}(\omega) = 0$$

clearly implies the result. Q.E.D.

$\Gamma \quad (\omega) = \left(\frac{\psi}{\sigma \cdot \sigma'}\right) \sim K \quad \Rightarrow \quad \omega$ is a meromorphic 2-form on S with polar curve $C+C'$.

$\Rightarrow$ By the global residue theorem, on P 656 & P33,

$$\sum_{p_\nu \in C \cap C'} \operatorname{Res}_{p_\nu}(\omega) = 0 .$$

Let $C \cap C' = \{ p_1, \cdots p_e \}$. Suppose D passes $p_2 \cdots p_e$.

At each p_ν, we have

$$\omega = \frac{g \, dz_1 \wedge dz_2}{f \; f'} \quad \text{around } p_\nu.$$

Here $(\underset{p_\nu}{f=0}) = C$, $(\underset{p_\nu}{f'=0}) = C'$, $(\underset{p_\nu}{g=0}) = D$.

and ψ defines D.