$\phi > 0$ in $(0,1)$, and define

$$\psi_m(x) = \phi(x-1) + \frac{1}{2}\phi(x-2) + \cdots + \frac{1}{m}\phi(x-m).$$

Then $\{\psi_m\}$ is a Cauchy sequence in the suggested top of $\mathcal{D}(R)$, but $\lim \psi_m$ does not have compact support, hence is not in $\mathcal{D}(R)$. "$\mathcal{D}(R)$ is not closed in $C^\infty(R)$." "If so, "$\mathcal{D}(R)$ should be complete since $C^\infty(\Omega)$ is complete."

Γ $0 < x-m < 1 \Rightarrow m < x < m+1, \Rightarrow \phi(x-m) > 0$

$\Rightarrow \lim \psi_m$ does not have compact support.

If $\phi(x) \leq M$,

$$\psi_{m+n}(x) - \psi_m(x) = \frac{1}{m+1}\phi(x-m-1) + \cdots + \frac{1}{m+n}\phi(x-m-n) \leq \frac{M}{m+1}$$

since

$$\begin{aligned}
&\phi(x-m-1) \neq 0 && \text{for } m+1 < x < m+2. \\
&\phi(x-m-2) \neq 0 && \text{for } m+2 < x < m+3 \\
&\quad\vdots && \quad\vdots \\
&\phi(x-m-n) \neq 0 && \text{for } m+n < x < m+n+1.
\end{aligned}$$

Thus $\{\psi_m\}$ is a Cauchy sequence for P_0.

Similarly, if $|\phi^\alpha(x)| \leq M_\alpha$,

$$\left|\psi^\alpha_{m+n}(x) - \psi^\alpha_m(x)\right| = \left|\frac{1}{m+1}\phi^\alpha(x-m-1) + \cdots + \frac{1}{m+n}\phi^\alpha(x-m-n)\right| \leq \frac{M_\alpha}{m+1}$$

since

$$\begin{aligned}
&\phi^\alpha(x-m-1) \neq 0 && \text{for } m+1 < x < m+2 \\
&\quad\vdots && \quad\vdots \\
&\phi^\alpha(x-m-n) \neq 0 && \text{for } m+n < x < m+n+1.
\end{aligned}$$

Again. point is here: $\exists\, l$ s.t if $\forall \epsilon > 0$

$$\sum_{N=l}^{\infty} \frac{2^{-N} P_N(\psi_m - \psi_{m+n})}{1 + P_N(\psi_m - \psi_{m+n})} < \frac{\epsilon}{2}.$$

$\lrcorner$