The integral is clearly continuous.

$$\frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=\epsilon}\frac{dh}{h} = \frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=\epsilon} d\log h \overset{h \text{ is holomorph}}{=} \frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=\epsilon}$$

$$\partial_{z_2}\log h \ \& \ \frac{1}{2\pi\sqrt{-1}}\int_{\{|z_2|\le\epsilon\}-\cup B_\delta(z_{2,i})} d(\partial_{z_2}\log h) = \frac{1}{2\pi\sqrt{-1}}\int_{\{|z_2|\le\epsilon\}-\cup B_\delta(z_{2,i})} \overline{\partial}_{z_2}\partial_{z_2}\log h = 0.$$

$\Rightarrow$ By Stokes' theorem,

$$\frac{1}{2\pi\sqrt{-1}}\int_{|z_2|=\epsilon}\frac{dh}{h}$$

$$= \frac{1}{2\pi\sqrt{-1}}\sum_i\int_{\partial B_\delta(z_{2,i})} \partial_{z_2}\log h \qquad \text{for each } z_1,\ 0<|z_1|\le\delta$$

where $z_{2,i}$'s are zeros of h in $|z_2|\le\epsilon$.

$$\frac{1}{2\pi\sqrt{-1}}\int_{\partial B_\delta(z_{2,i})}\frac{\frac{\partial h}{\partial z_2}dz_2}{h} \quad \text{is an integer} \qquad \text{for,}$$

if $h(z_1,z_2) = (z_2 - z_{2i})^\ell g(z_1,z_2)$, $g(z_1, z_{2i}) \neq 0$,

then
$$\frac{\frac{\partial h}{\partial z_2}}{h} = \frac{\ell(z_2-z_{2i})^{\ell-1}g + (z_2-z_{2i})^\ell\frac{\partial g}{\partial z_2}}{(z_2-z_{2i})^\ell g}$$

$$= \frac{\ell}{z_2 - z_{2i}} + \underbrace{f(z_1,z_2)}_{\text{holomorphic.}},$$

$$\Rightarrow \frac{1}{2\pi\sqrt{-1}}\int_{\partial B_\delta(z_{2i})}\frac{\frac{\partial h}{\partial z_2}dz_2}{h} = \frac{1}{2\pi\sqrt{-1}}\int_{\partial B_\delta(z_{2i})}\frac{\ell}{z_2-z_{2i}}dz_2$$

$$= \frac{1}{2\pi\sqrt{-1}}\int_0^{2\pi}\frac{\ell}{\cancel{\delta}\,\cancel{e^{i\theta}}}\cancel{\delta}\, i\, \cancel{e^{i\theta}}\,d\theta = \ell.$$

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