Note that $z_2^{d'} + a_1(f(z_1,z_2))\, z_2^{d'-1} + \cdots + a_{d'}(f(z_1,z_2)) \equiv 0$ for all (z_1, z_2). For generic w, $\exists$ at most d' distinct z_2's satisfying the polynomial — ✩

$$z_2^{d'} + a_1(w)\, z_2^{d'-1} + \cdots + a_{d'}(w) = 0 \quad \text{--- (*)}$$

$\Rightarrow$ For the fixed generic w, we have d distinct $(z_1, z_2)'$s s.t. $f(z_1, z_2) = w$. and all z_2's are distinct. Here we assumed $f(z_1,z_2)-w$ does not contain z_2-axis.

$\Rightarrow$ We have d distinct z_2's satisfying (*) which contradicts to the above ✩. $\lrcorner$

As a corollary, we have the following special case of the nullstellensatz:

If $h(z) \in \mathcal{O}_z$. vanishes at $z=0$, then

$$h(z)^d \in \{f_1(z), \cdots f_n(z)\}.$$

Proof. As $z \to 0$, both $f(z)$ and $z_\nu(f(z)) \to 0$. ∵ Continuity & Weierstrass preparction theorem imply these. $\lrcorner$

Consequently, the coefficients $a_\nu(w)$ in the polynomial H vanish at $w=0$.

∵ Since $a_\nu(w) = a_\nu(f(z)) = \sum h(z_{i_1}(f(z)))h(z_{i_2}(f(z)))\cdots h(z_{i_\nu}(f(z)))$, at $w=0$, $a_\nu(w) = 0$ $\lrcorner$

This implies that $h^d \equiv 0$ modulo $\{f_1, \cdots f_n\}$.

∵ $h^d(z) + a_1(f(z))\, h^{d-1}(z) + \cdots + a_d(f(z)) \equiv 0$

$\Rightarrow h^d(z) = -a_1(f(z))\, h^{d-1} - \cdots - a_d(f(z))$