$$= X_0 \sum_{i=0}^{n} \frac{\partial F_\alpha}{\partial X_i}(p)\,(x_i - \alpha_i) = X_0 \sum_{i=1}^{n} X_0^{d-1} \frac{\partial f_\alpha}{\partial x_i}(p)\,(x_i - \alpha_i)$$

$$= X_0^{d} \sum_{i=1}^{n} \frac{\partial f_\alpha}{\partial x_i}(p)\,(x_i - \alpha_i) \qquad \text{since } x_0 = \frac{X_0}{X_0} = 1 = \alpha_0 .$$

I think $\frac{1}{d} \sum_{i=0}^{n} \frac{\partial F}{\partial X_i} = F$ is useless and wrong

Example. $F = X_0^2 + X_1 X_0$

$$\frac{\partial F}{\partial X_0} = 2X_0 + X_1, \qquad \frac{\partial F}{\partial X_1} = X_0 \Rightarrow \frac{1}{2}(2X_0 + X_1 + X_0) \neq F .$$

In a similar way we may define the tangent cone to a variety $V \subset \mathbb{P}^n$ at a (possibly singular) point $p \in V$. First, if $V$ is a hypersurface cut out by the homogeneous polynomial $F$, and $p$ a point of multiplicity $k$ on $V$ — so that all the partial derivatives of $F$ of order $\leq k-1$ vanish — we take the tangent cone to $V$ at $p$ to be the locus

$$T_p(V) = \sum \frac{\partial^k F}{\partial^{i_0} X_0 \cdots \partial^{i_n} X_n}(p) \cdot X_0^{i_0} \cdots X_n^{i_n} = 0 .$$

In general, we will take the tangent cone to a variety $V \subset \mathbb{P}^n$ at a point $p$ to be the intersection of the tangent cones at $p$ to all the hypersurfaces containing $V$ near $p$. This may be realized alternatively as the union of the tangent lines at $p$ to all curves lying on $V$ and passing through $p$; or as the limiting position of chords $\lim_{\lambda \to 0} \overline{p, q(\lambda)}$ where $q(\lambda)$ is an arc in $V$ with $q(0) = p$.

See P220, Theorem 4B (a), (d), Whitney. Complex Analytic Varieties