$$f(g) = P(gA_1g^{-1}, \cdots gA_kg^{-1})$$
$$= P((I+g')A_1(I-g'), \cdots (I+g')A_k(I-g')) + [\partial]$$
$$= P(A_1, \cdots A_k) + \sum_i P(A_1, \cdots g'A_i - A_ig', \cdots A_k) + [\partial].$$

$\ulcorner$ $f(g) = P(gA_1g^{-1}, \cdots gA_kg^{-1}) = P((I+g')A_1(I-g'+[\partial]), \cdots$
$(I+g')A_k(I-g'+[\partial])) = P((I+g')A_1(I-g'), \cdots (I+g')A_k(I-g'))$
$+ P((I+g')A_1[\partial], \cdots)$
$= P((I+g')A_1(I-g') \cdots (I+g')A_k(I-g')) + [\partial]$
$= P((A_1+g'A_1)(I-g'), \cdots (A_k+g'A_k)(I-g')) + [\partial]$
$= P(A_1 - A_1g' + g'A_1 - g'A_1g', \cdots) + [\partial]$
$= P(A_1, \cdots A_k) + \sum_i P(A_1, \cdots g'A_i - A_ig' \cdots A_k) +$
$P(g'A_1 - A_1g', g'A_2 - A_2g', \cdots) + \cdots + [\partial].$

$g_{ij}'^2$ appears.

$= P(A_1 \cdots A_k) + \sum_i P(A_1 \cdots g'A_i - A_ig', \cdots A_k) + [\partial]$

Notice that if any entry of $g_{ij}'^2$'s appears, then we have the zero derivative. this is the point. ⌟

But if P is invariant, $f = P(A_1, \cdots A_k)$; thus all higher-order terms in the power series for f vanish, and in particular
$$\sum_i P(A_1, \cdots, g'A_i - A_ig', \cdots A_k) = 0.$$

$\ulcorner$ If P is invariant, $f = P(A_1 \cdots A_k) = P(A_1 \cdots A_k)$
$+ \sum_i P(A_1, \cdots g'A_i - A_ig', \cdots A_k) + [\partial] \cdots$ (*)

$\Rightarrow \sum_i P(A_1, \cdots g'A_i - A_ig', \cdots A_k) = 0$ & $[\partial] = 0.$ ⌟