$p+q$ even, $\Rightarrow p+q$ = even $\xrightarrow{=0}$ by $Q(H^{p,q}, H^{p',q'})=0$ unless $p=q'$, $q=p'$

$$Q(\zeta+\bar\zeta,\ \zeta+\bar\zeta) = \cancel{Q(\zeta,\zeta)} + \cancel{Q(\bar\zeta,\bar\zeta)} + Q(\zeta,\bar\zeta)+Q(\bar\zeta,\zeta)$$
$$= Q(\zeta,\bar\zeta)+Q(\bar\zeta,\zeta)$$

$$\Rightarrow\ Q(\zeta,\bar\zeta) = \int_M \zeta\wedge\bar\zeta\wedge\omega^k = \int_M (-1)^{\text{even}}\,\bar\zeta\wedge\zeta\wedge\omega^k = Q(\bar\zeta,\zeta).$$

$\Rightarrow\ Q(\zeta+\bar\zeta,\ \zeta+\bar\zeta) = 2\,Q(\zeta,\bar\zeta) > 0$ by Hodge–Riemann relations.

$\Rightarrow\ Q(\zeta+\bar\zeta,\ \zeta+\bar\zeta) > 0$

$p+q$ = odd, Q is non degenerate skew-symmetric form on $P^{p+q}(M)$. Let $\eta, \zeta \in P^{p+q}(M)$.

$$Q(\eta,\zeta) = \int_M \eta\wedge\zeta\wedge\omega^k = \int_M (-1)^{(p+q)^2}\zeta\wedge\eta\wedge\omega^k = -\int_M \zeta\wedge\eta\wedge\omega^k$$

$= -Q(\zeta,\eta) \quad\Rightarrow\quad Q$ is skew-symmetric

Suppose $Q(\zeta,\eta)=0$ for all $\eta \in P^{p+q}(M)$,
$\Rightarrow\ Q(\zeta,\zeta)=0 \iff \zeta=0$ by Hodge–Riemann relations.
$\Rightarrow\ Q$ is non-degerate on $P^{p+q}(M)$.

In either case, since we have the Lefschetz decomposition

$$H^m = \oplus L^k P^{m-2k}$$

and $Q(L^k\zeta, L^k\eta) = Q(\zeta,\eta)$; the bilinear relations tell us that Q is non-degenerate on $H^{n-k}(M)$.

$$\ulcorner\ Q(L^k\zeta, L^k\eta) = \int_M L^k\zeta\wedge L^k\eta\wedge\omega^k,\quad \zeta,\eta\in H^{n-3k}$$
$$= \int_M \zeta\wedge\omega^k\wedge\eta\wedge\omega^k\wedge\omega^k = \int_M \zeta\wedge\eta\wedge\omega^{3k}$$
$$= Q(\zeta,\eta)$$