$f_x = f|_{E_x} : E_x \longrightarrow F_x$ is linear. Note that

$$\ker(f) = \bigcup \ker f_x \subset E \quad \text{and} \quad \mathrm{Im}(f) = \bigcup \mathrm{Im} f_x \subset F$$

are subbundles of E and F, respectively $\Leftrightarrow$ the maps f_x are all have the same rank.

⌈

$$\begin{tikzcd} E_U \arrow[r, "\varphi_U"] \arrow[d, "f_U"'] & U \times \mathbb{C}^n \arrow[d, "f_U"] \\ F_U \arrow[r, "\psi_U"] & U \times \mathbb{C}^m \end{tikzcd}$$

induced by f_U.

Assume $n \geq m$

$\Rightarrow$ We have only to consider $U \times \mathbb{C}^n \xrightarrow{f_U} U \times \mathbb{C}^m$.

$\Rightarrow$ $U \times \mathbb{C}^n \xrightarrow{f_U} U \times \mathbb{C}^m$ can be expressed as

$$(x, z_1, z_2 \cdots z_n) \longmapsto \left(x, \begin{pmatrix} a_{11}(x), & \cdots & a_{1n}(x) \\ a_{21}(x) & \cdots & a_{2n}(x) \\ \vdots & & \vdots \\ a_{m1}(x) & \cdots & a_{mn}(x) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}\right)$$

$\mathrm{rank}(a_{ij}(x)) = k$

For a fixed point $x_0 \in U$, by changing a basis for $\mathbb{C}^m$, (inducing a diffeomorph of $U \times \mathbb{C}^m$)

we have
$$\begin{pmatrix} a_{11}(x_0) & \cdots & a_{1n}(x_0) \\ a_{21}(x_0) & \cdots & a_{2n}(x_0) \\ \vdots & & \\ a_{m1}(x_0) & \cdots & a_{mn}(x_0) \end{pmatrix} = \begin{array}{c} \\ k \\ \\ \\ \end{array} \begin{pmatrix} 1, & 0, & \cdots & & & 0 \\ 0 & 1, & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ 0 & 0 & \cdots & 1 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider a map F from $U \times \mathbb{C}^n \longmapsto U \times \mathbb{C}^{m+n-m}$ given by
$\mathbb{C}^k \subset U \times \mathbb{C}^n$, $\quad \mathbb{C}^{n-k} \subset U \times \mathbb{C}^{m+n-m}$

$$F(x, z) = (x, \pi \circ f_U(x, z), z_{k+1}, z_{k+2}, \cdots z_n).$$

where $\pi : U \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$.

Then since $dF(x_0, z) =$ identity for all z, F is a diffeomorphism.