fact

$$\overline{P_L, q'} = \overline{P_{L'}, q'}$$

i.e., $q = q'$

is the singular point of C_L.

$\ulcorner$ $P_L, P_{L'} \in h_L$, and since q' is a double point, $\overline{P_L, q'}$ and $\overline{P_{L'}, q'}$ are tangent to S at q'.
If $\overline{P_L, q'} = \overline{P_{L'}, q'}$, then $\overline{P_L, P_{L'}} \ni q'$. But since we proved that $\overline{P_L, P_{L'}}$ is tangent to S at q, $q = q'$
($\because$ $\overline{P_L, P_{L'}}$ is tangent to S at q', too.)
$\lrcorner$

This is clear: if $\overline{P_L, q'}$ and $\overline{P_{L'}, q'}$ were distinct and nonsingular, then they would necessarily lie in one special pencil, which could only be the pencil of lines through q' in h_L — but L and L' are the only pencils of X in h_L.

$\ulcorner$ Since $\overline{P_L, q'}$ & $\overline{P_{L'}, q'}$ are nonsingular, they are in special pencils by P791. Let $\overline{P_L, q'} \in L_1$ and $\overline{P_{L'}, q'} \in L_2$, L_1 & L_2 special. If $L_1 = \sigma(p, h)$, then, since L_1 is special, by the definition on P793, $\overline{P_L, q'}$ must be tangent to S at p. $\Rightarrow$ ① $p \neq P_L, q'$
$\Rightarrow$ Since $\#(\overline{P_L, q'} \cdot S) \geq 5$, it is impossible.
② $p = P_L$. $\Rightarrow$ $L_1 = \sigma(P_L, h)$ $\Rightarrow$ $\overline{P_L, q'} \in \sigma(P_L, h) \cup \sigma(P_L, h_L)$
③ $p = q'$ $\Rightarrow$ $L_1 = \sigma(q', h)$. $\Rightarrow$ $\overline{P_L, q'}$ is singular $\Rightarrow$ $*$