Since $df = df' \Leftrightarrow f = f' + \lambda,\ \lambda \in \mathbb{C}$, we see that the dimension of $H^0(S, \mathcal{O}(D))$ is one more than the dimension of the vector space V of differentials of the second kind holomorphic on $S - \{p_\lambda\}$ with no periods and poles of order ≤ 2 at p_λ.

Γ $d(f-f') = 0 \Leftrightarrow f - f' = \lambda \in \mathbb{C}$.

$$\begin{tikzcd} \mathcal{L}(D) \arrow[r, "\phi"] & V \\ f \arrow[r, mapsto] & df \end{tikzcd} \Rightarrow \phi \text{ is onto by the argument above, and } \ker\phi = \mathbb{C}.$$

$\Rightarrow \dim \mathcal{L}(D) = \dim V + 1.$ $\lrcorner$

By the Kodaira vanishing theorem, for any $p \in S$

$$H^1(S, \Omega^1(p)) = 0,$$

and so from the exact sequence

$$0 \longrightarrow \Omega^1(p) \longrightarrow \Omega^1(2p) \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

we see that there exists a meromorphic form on S, holomorphic on $S - \{p\}$ and having a double pole at p; clearly this can not have any residues.

Γ $H^0(S, \Omega^1(p)) \longrightarrow H^0(S, \Omega^1(2p)) \longrightarrow \mathbb{C}_p \longrightarrow 0 = H^1(S, \Omega^1(p))$

$\Rightarrow \exists$ a section $\sigma \in H^0(S, \Omega^1(2p))$ s.t. $\sigma(p) \neq 0$.

Since we have a section s_0 of $[2p]$ s.t. $(s_0 = 0) = 2p$,

consider σ/s_0. $\Rightarrow$ σ/s_0 is a form on S which is meromorphic. Obviously, σ/s_0 has a double pole at p, and holomorphic on $S - \{p\}$. This σ/s_0 can not have residue at p.