isomorphic, and it follows that any two smooth quadric surfaces in $\mathbb{P}^3$ are projectively isomorphic.

Γ Let Q be a symmetric 4×4 matrix of complex numbers. Consider the map $Q : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$ defined by
$$\begin{array}{ccc} \mathbb{C}^4 & \longrightarrow & \mathbb{C}^4 \\ \Psi & & \\ X & \longmapsto & QX. \end{array}$$

$\Rightarrow \exists$ a nonzero eigenvector $v_1 \in \mathbb{C}^4$ with $Qv_1 = \lambda_1 v_1$.

Consider the following map $\phi : \mathbb{C}^4 \longrightarrow \mathbb{C}$ defined by
$$\begin{array}{ccc} \mathbb{C}^4 & \longrightarrow & \mathbb{C} \\ \Psi & & \\ X & \longmapsto & {}^tX v_1. \end{array}$$

$\Rightarrow \quad 0 \longrightarrow \ker\phi \longrightarrow \mathbb{C}^4 \longrightarrow \mathbb{C} \longrightarrow 0.$

$\Rightarrow \quad Q : \ker\phi \longrightarrow \ker\phi$, since, if $Y \in \ker\phi$,

$$\phi(QY) = {}^t(QY)v_1 = {}^tY\,{}^tQ v_1 = {}^tYQv_1 = {}^tY(\lambda_1 v_1)$$
$$= \lambda_1\, {}^tYv_1 = \lambda_1 \phi(Y) = 0.$$

eigenvectors

$\Rightarrow$ By the induction on dim, we have v_1, v_2, v_3, v_4 s.t ${}^tv_i v_j = 0$, and $Qv_i = \lambda_i v_i$.

$\Rightarrow \exists$ a nonsingular matrix A s.t
$$A^{-1}QA = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

It remains to prove that $A^{-1} = {}^tA$, to show that
$${}^tUQU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{ since } \begin{pmatrix} \lambda_1^{\frac12} & 0 & 0 & 0 \\ 0 & \lambda_2^{\frac12} & 0 & 0 \\ 0 & 0 & \lambda_3^{\frac12} & 0 \\ 0 & 0 & 0 & \lambda_4^{\frac12} \end{pmatrix} A^{-1}QA \begin{pmatrix} \lambda_1^{\frac12} & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$= I$ in case Q is nonsingular $\Leftrightarrow \lambda_i \neq 0$ for all i.