Γ $\omega = 0 \iff r = \dim|\mathcal{J}_{P_0}(n)| = \frac{n(n+3)}{2} - d.$

Note that the linearly independent d conditions are given on $H^0(\mathbb{P}^2, \mathcal{O}(nH)) \iff$ the dimension of the resulting space is $\dim H^0(\mathbb{P}^2, \mathcal{O}(nH)) - d$.

$\Rightarrow \dim H^0(\mathbb{P}^2, \mathcal{O}(nH - C_0)) = \dim|\mathcal{J}_{P_0}(n)| + 1 = r + 1 = \frac{(n+2)(n+1)}{2} - d$

$= \dim H^0(\mathbb{P}^2, \mathcal{O}(nH)) - d.$

$\Gamma_0 = \{p \in C \cap D\} \Rightarrow \deg \Gamma_0 = mn.$

$\Rightarrow$ For each $p \in \Gamma_0$, $\exists$ a linear condition on $H^0(\mathbb{P}^2, \mathcal{O}(m+n-3))$

i.e $\sigma(p) = 0 = a_1 \sigma_1(p) + \cdots + a_{\binom{m+n-1}{2}} \sigma_{\binom{m+n-1}{2}}(p)$

But we only need $(mn-1)$ condition, since the last one is satisfied by Cayley-Bacharach, i.e $\sigma(P_{mn}) = 0$.

$\Rightarrow$ mn points do not give linearly independent mn conditions $\Rightarrow \omega \neq 0 \Rightarrow h^1(\mathcal{J}_{P_0}(n)) \neq 0$ $\lrcorner$

· Returning now to a general surface S with line bundle $L \to S$, we will show: Suppose that $\dim|L| > 0$ and that the complete linear system $|L|$ has no base curves. Assume moreover that the irregularity $q = 0$. Then the superabundance $\omega = h^1(L)$ is given by

$$\omega = \dim|\mathcal{J}_P(K_S + L)| - 2p_g + \dim|K_S - L| + 2,$$

where $P = C \cdot C'$ for general curves $C, C' \in |L|$.

Proof. Let $s, s' \in H^0(\mathcal{O}_S(L))$ define C, C', respectively, and consider the Koszul complex