Linear Systems of Quadrics

Thus far, we have examined the geometry of a single quadric hypersurface in $\mathbb{P}^n$. We would now like to consider linear systems of quadrics; specifically we will study linear systems of quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$.

We begin with $\mathbb{P}^2$. In the complete system $W \cong \mathbb{P}^5$ of conic plane curves, let $W_1 \subset W$ be the subvariety of conics of rank two or less and $W_2 \subset W_1$ the set of conics of rank one.

Pf $\dim H^0(\mathbb{P}^2, \mathcal{O}(2H)) = \binom{2+2}{2} = {}_4C_2 = 6$

$\Rightarrow \dim |2H| = 5 = \dim \mathbb{P}^5$

$W = \{ Q \mid Q \text{ 3x3 symmetric matrix} \} / \mathbb{C}^* = \mathbb{P}^5$

$\Rightarrow$ Since $\alpha^3 D(Q) = D(\alpha Q) = \text{Determinat}(\alpha Q)$,

$W_1 = \{ Q \mid D(Q) = 0, \ Q \text{ 3x3 symmetric matrix} \} / \mathbb{C}^*$.

$\Rightarrow$ W_1 is a subvariety of W. $\square$

W_1 is a hypersurface in W; we first ask for its degree.

Pf By the argument above, $W_1 = \{ D(Q) = 0 \}$ in W

$\Rightarrow$ W is a hypersurface. $\square$

This question may be answered in four ways: