we have found a whole collection of "new" vector bundles over $\mathbb{P}^2$.

⊩ By the claim, $\dim H^0(S, \mathcal{E}) = 1$.
Suppose $E = L \oplus L'$. $\Rightarrow H^0(S, \mathcal{O}(E)) = H^0(S, \mathcal{O}(L \oplus L')) = H^0(S, \mathcal{O}(L) \oplus \mathcal{O}(L')) = H^0(S, \mathcal{O}(L)) \oplus H^0(S, \mathcal{O}(L')) \ni s = (s_1, s_2)$.
$\Rightarrow s_1 \not\equiv 0 \quad s_2 \not\equiv 0$ since otherwise $\dim(s=0) \geq 1$.
For example, $\mathbb{C}^2 \xrightarrow{s} \mathbb{C}^2$

$(x, y) \longmapsto (x, 0) \quad \Rightarrow \quad x = 0. \quad (s=0) = \{(0, y)\}$.

The "new" v. b. is not a type of $L \oplus L'$ of two line bundles. L above is not $\mathcal{L} = \mathcal{O}$. ⅃

Proof. If $s' \in H^0(\mathbb{P}^2, \mathcal{O}(E))$ also defines Z, then $s \wedge s' \in H^0(\mathbb{P}^2, \mathcal{O}(\Lambda^2 E)) = H^0(\mathbb{P}^2, \mathcal{O}) = \mathbb{C}$, since $\Lambda^2 E$ is a trivial bundle because $c_1(E) = 0$.

⊩ $c_1(E) = -c_1(\mathcal{O}) = 0$ since $\mathcal{O}$ is a trivial bundle. By p145, $\Lambda^2 E$ is trivial. ⅃
"I think" $c_1(E)$ is redundant.

Thus either $s \wedge s' \equiv 0$, in which case s' is a constant multiple of s, or else $s \wedge s'$ is nowhere zero, which is excluded by the assumption that Z is non-empty.

⊩ $s \wedge s'$ is a constant holomorphic function on $\mathbb{P}^2$.
$\Rightarrow s \wedge s' \equiv 0$ or $s \wedge s'$ is nowhere zero, which is