this is equivalent to the assertion that for any k-cycle $A$ on $M$, there exists a closed $(n-k)$-form $\varphi$ s.t

for any $(n-k)$ cycle $B$ on $M$,

$$\int_B \varphi = \#(A \cdot B).$$

Let $\varphi$, $\psi$ closed forms on the oriented manifold $M$. Then $\varphi \wedge \psi$ is closed and by virtue of the relation

$$\varphi \wedge (\psi + d\eta) = \varphi \wedge \psi + (-1)^{\deg \varphi} d(\varphi \wedge \eta)$$

we see that the de Rham class of $\varphi \wedge \psi$ depends only on the de Rham classes of $\varphi$ and $\psi$.

$\Rightarrow$ We have bilinear maps $H^k_{DR}(M) \otimes H^{k'}_{DR}(M) \longrightarrow H^{k+k'}_{DR}(M)$

and in particular a pairing

$$H^k_{DR}(M) \otimes H^{n-k}_{DR}(M) \longrightarrow H^n_{DR}(M) \cong \mathbb{C}$$

We will now relate this pairing in de Rham cohomology to the intersection of cycles via Poincare duality; to do this we must first establish the Künneth formula.

Suppose $M = \{\sigma^k_\alpha\}_{\alpha,k}$ and $N = \{\sigma'^k_\alpha\}_{\alpha,k}$ are two simplicial complexes.
See page Th 23.16 Maunder. # P 338, Munkres
The products $\sigma^k_\alpha \times \sigma'^l_\beta$ give a cell-decomposition of the product space $M \times N$, with boundary operator

$$\partial(\sigma^k_\alpha \times \sigma'^l_\beta) = \partial\sigma^k_\alpha \times \sigma'^l_\beta + (-1)^k \sigma^k_\alpha \times \partial\sigma'^l_\beta.$$