To prove assertion 2, we have to show that, for Θ a curvature matrix for the bundle $[D]$, the cohomology class $\frac{i}{2\pi}\Theta$ is the Poincare dual of $(D) = \sum a_i(V_i)$ – i.e.

that for every real closed form $\psi \in A^{2n-2}(M)$,

$$\frac{i}{2\pi}\int_M \Theta \wedge \psi = \sum a_i \int_{V_i} \psi .$$

Γ $$\sum a_i \int_{V_i} \psi = \sum a_i \int_{(V_i)} \psi = \int_{\sum a_i (V_i)} \psi = \#(\sigma_\psi \cdot \sum a_i (V_i))$$

$$= \int_M \psi \wedge \frac{i}{2\pi}\Theta$$ see p59 $\rfloor\rfloor$

Since both $D \longmapsto c_1([D])$ and $D \longmapsto \eta_D$ are homomorphisms from $Div(M)$ to $H^2_{DR}(M)$, we may take $D = V$ an irreducible subvariety.

First, we compute the curvature form of a metric connection on $[D]$. To do this, let e be a local non-zero holomph section of $[V]$ and write

$$|e(z)|^2 = h(z).$$

Then for any section $s = \lambda \cdot e$, the connection matrix θ for the metric connection D in terms of the frame e must satisfy

by the def of metric connection

$$\theta = \theta^{1,0}. \quad (De = D'e + D''e = D'e + \bar{\partial} e = \theta^{1,0} e + 0 = \theta e)$$

and

$$d\langle s,s\rangle = d|s|^2 = (Ds, s) + \langle s, Ds\rangle = \langle (d\lambda + \theta\lambda) e, \lambda e\rangle$$
$$+ \langle \lambda e, (d\lambda + \theta\lambda) e\rangle = h\, d\lambda\, \bar{\lambda} + |\lambda|^2 \theta h + \lambda \overline{d\lambda}\, h$$
$$+ h\bar{\theta}|\lambda|^2 = h\bar{\lambda}\, d\lambda + h\lambda\, d\bar{\lambda} + h|\lambda|^2(\theta + \bar{\theta}).$$