97.1.8

the de Rham isomorphism. See P44 and P40

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{A}^0 \xrightarrow{d} Z^1 \longrightarrow 0.$$

$\Rightarrow$ We have $H^1(M, \mathbb{C}) \cong H^0(M, Z^1) \big/ d H^0(M, \mathcal{A}^0)$.

$$\begin{tikzcd}
0 \arrow[r] & C^0(\underline{U}, \mathbb{C}) \arrow[r] \arrow[d, "\delta"] & C^0(\underline{U}, \mathcal{A}^0) \arrow[r, "d"] \arrow[d, "\delta"] & C^0(\underline{U}, Z^1) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^1(\underline{U}, \mathbb{C}) \arrow[r] & C^1(\underline{U}, \mathcal{A}^0) \arrow[r, "d"] & C^1(\underline{U}, Z^1) \arrow[r] & 0
\end{tikzcd}$$

$$f \longmapsto \omega = df \longmapsto 0 \in C^1(\underline{U}, Z^1)$$

$$z \longmapsto \delta f \longmapsto 0$$

$\Rightarrow \exists\ f_\alpha \in A^0(U_\alpha)$ s.t $z = \delta f$, $f \in C^0(\underline{U}, \mathcal{A}^0)$ i.e.,

$$z_{\alpha\beta} = f_\beta - f_\alpha.$$

$\Rightarrow$ Since $z_{\alpha\beta}$ is constant, $df_\beta - df_\alpha = 0$.

$\Rightarrow$ $df_\alpha = df_\beta$ represents the image of z in $H^1_{DR}(M, \mathbb{C})$. On the other hand, take the image of $\iota_1^* z$ under the Dolbeault isomorphism: we write

$$z_{\alpha\beta} = f_\beta - f_\alpha$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar\partial} Z^{0,1}_{\bar\partial} \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & C^0(\underline{U}, \mathcal{O}) \arrow[r] \arrow[d, "\delta"] & C^0(\underline{U}, \mathcal{A}^{0,0}) \arrow[r, "\bar\partial"] \arrow[d, "\delta"] & C^0(\underline{U}, Z^{0,1}_{\bar\partial}) \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & C^1(\underline{U}, \mathcal{O}) \arrow[r] & C^1(\underline{U}, \mathcal{A}^{0,0}) \arrow[r] & C^1(\underline{U}, Z^{0,1}_{\bar\partial}) \arrow[r] & 0
\end{tikzcd}$$

$$f \longmapsto \bar\partial f$$

$$z \longmapsto \delta f$$

$$\Rightarrow \quad H^1(M, \mathcal{O}) \xrightarrow{\cong} H^{0,1}_{\bar\partial}(M)$$

$$[z] \longmapsto [\bar\partial f]$$

Note: If there is another g s.t $\delta g = \delta f$, then since $g - f \in A^{0,0}(M)$, $[\bar\partial f] = [\bar\partial g] \in H^{0,1}_{\bar\partial}(M)$ $= \dfrac{\operatorname{Ker}\bar\partial}{\bar\partial A^{0,0}(M)}$.