Then $\varphi - d\eta \equiv 0 \ (du, dv_1, \cdots dv_{\ell-1})$.

⌈ Let $\eta = \sum \eta_I \, dx_I$.

$$\Rightarrow d\eta = \sum \frac{\partial \eta_I}{\partial v_\ell} dv_\ell \wedge dx_I + \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

$$+ \sum_{j>\ell} \cancel{\frac{\partial \eta_I}{\partial v_j}}^{=0} dv_j \wedge dx_I.$$

$$= \pm \sum \frac{\partial \eta_I}{\partial v_\ell} dx_I \wedge dv_\ell + \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

$$= \pm \frac{\partial \eta}{\partial v_\ell} \wedge dv_\ell + \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

$$= \pm \varphi'' \wedge dv_\ell + \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

$$= \pm (\varphi - \varphi') + \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

We may arrange the sign so that

$$d\eta = \varphi - \varphi' + \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

$$\Rightarrow \quad \varphi - d\eta = \varphi' - \sum_{j<\ell} \frac{\partial \eta_I}{\partial v_j} dv_j \wedge dx_I$$

$$\equiv 0 \ (du, dv_1, \cdots dv_{\ell-1}).$$

where $dx_I = du, dv_1, \ \ldots dv_{\ell-1}$, as we see above ⌋

Continuing in this way, we may assume that $\varphi \equiv 0 \ (du)$. Then $d\varphi = 0 \Rightarrow (\partial\varphi/\partial v_j) = 0$, and so the v's may effectively be ignored.