on $\mathbb{P}^n$ around p_i, we have

$$\pi_*\left(X_j \frac{\partial}{\partial X_j} \right) = x_j \frac{\partial}{\partial x_j}, \quad j \neq i,$$

and

$$\pi_*\left(X_i \frac{\partial}{\partial X_i} \right) = -\sum_{j \neq i} x_j \frac{\partial}{\partial x_j}.$$

$\ulcorner$ $v(X) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i} = 0$

$\Rightarrow \quad \alpha_i X_i \pi_* \frac{\partial}{\partial X_i} = 0 = \sum_{i=0}^{n} (\alpha_i - \alpha_0) X_i \pi_* \frac{\partial}{\partial X_i}$

$\Rightarrow \quad (\alpha_i - \alpha_0) X_i = 0 \quad \Rightarrow \quad \alpha_i = \alpha_0$ or $X_i = 0$.

① $\alpha_i = \alpha_0 \Rightarrow$ contradiction to the assumption that α_i's are all distinct nonzero complex numbers.

② $X_i = 0 \quad i = 1, \cdots n$

$\Rightarrow X_0 \neq 0. \quad \Rightarrow \quad P_0 = [1, 0 \cdots 0]$

Similarly, we get $v = 0$ at $P_i = [\underline{0 \cdots} \underset{i}{1} \cdots 0]$.

$v(p_i) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i} = \pi_*\left(\alpha_i \frac{\partial}{\partial X_i}\right)$

$= \alpha_i \pi_* \frac{\partial}{\partial X_i} = -\alpha_i \sum_{j \neq i} \frac{X_j}{X_i^2} \cdot \frac{\partial}{\partial x_j}$ but since $X_j = 0, j \neq i$

$\Rightarrow v(p_i) = 0.$

$\pi_*\left(X_j \frac{\partial}{\partial X_j} \right) = X_j \pi_*\left(\frac{\partial}{\partial X_j}\right) = X_j \frac{1}{X_i} \frac{\partial}{\partial x_j} = \overset{\frac{X_j}{X_i}}{\vee} \frac{\partial}{\partial x_j} \quad j \neq i$

$= x_j \frac{\partial}{\partial x_j}.$

$\pi_*\left(X_i \frac{\partial}{\partial X_i}\right) = X_i \left(-\sum_{j \neq i} \frac{X_j}{X_i^2} \frac{\partial}{\partial x_j} \right) = -\sum_{j \neq i} x_j \frac{\partial}{\partial x_j} \quad j = i.$ $\lrcorner$