We now come to the proof of the Garding inequality, where we assume the Weitzenböck in the form

$$(\Delta\psi)_{I\bar{J}} = -\sum_K \psi_{I\bar{J}\bar{K}K} + A'(\psi)_{I\bar{J}} .$$

Inequalities of the type

$2\alpha\beta \leq \epsilon\alpha^2 + \frac{1}{\epsilon}\beta^2$ will be used repeatedly,

and $\Phi = \Phi' \wedge \overline{\Phi'}$ denotes the volume form.

$$\eta = C\,((\bar\nabla\psi, \psi) \wedge \omega^{n-1})$$
$$\equiv \quad -\sum_{I,J,K} (-1)^{k-1}\, \psi_{I\bar{J},\bar{K}}\, \overline{\psi_{I\bar{J}}}\, \varphi_1 \wedge \cdots \wedge \hat{\varphi}_K \wedge \cdots \wedge \varphi_n) \wedge \overline{\Phi'}$$

$$\Gamma\ \psi = \sum_{I,J} \psi_{I\bar{J}}\ \varphi_I \wedge \overline{\varphi_J}$$

$$\bar\nabla\psi = \sum_{IJ} \nabla''(\psi_{I\bar{J}}\ \varphi_I \wedge \overline{\varphi_J}) \equiv \sum_{IJ} \bar\partial\psi_{I\bar{J}} \otimes \varphi_I \wedge \overline{\varphi_J} + (A''(\psi))\,\mathbb{D}$$

$$= \sum_{IJ,K} \psi_{I\bar{J},\bar{K}}\ \overline{\varphi_K} \otimes \varphi_I \wedge \overline{\varphi_J}$$

$$\Rightarrow \langle\bar\nabla\psi, \psi\rangle \equiv \sum_{IJ,K} \psi_{I\bar{J}.\bar{K}}\ \overline{\psi_{I\bar{J}}}\ \overline{\varphi_K}\ \ (+\text{ involving } \psi_{I\bar{J}})$$

$$\Rightarrow \langle\bar\nabla\psi, \psi\rangle \wedge \omega^{n-1} \equiv C_1 \sum_{IJ,K} \psi_{I\bar{J}.\bar{K}}\ \overline{\psi_{I\bar{J}}}\ \overline{\varphi_K} \wedge \varphi_1 \wedge \overline{\varphi_1} \wedge \cdots \wedge \widehat{\varphi_K \wedge \overline{\varphi_K}} \wedge \cdots \wedge \varphi_n \wedge \overline{\varphi_n}.$$

$$= C'\,(-\sum_{IJ,K} \psi_{I\bar{J},\bar{K}}\ \overline{\psi_{I\bar{J}}}\ \varphi_1 \wedge \cdots \wedge \hat{\varphi}_K \cdots \wedge \varphi_n \wedge \overline{\varphi_1} \wedge \cdots \overline{\varphi_n}) \quad C' \neq 0$$

The expression shows that η is globally defined, and since it has type $(n-1, n)$, $d\eta = \partial\eta$.
By Stoke's theorem,

$$\int_M \partial\eta = 0.$$