$$\begin{tikzcd} E_2 \arrow[r] \arrow[d] & E_1 \arrow[d, "\Phi_1-\Psi_1-\alpha_0\circ\partial_N"] \arrow[dl, dashed, "\alpha_1"'] & \\ F_2 \arrow[r] & \operatorname{Ker}\partial_N \arrow[r] & 0 \end{tikzcd}$$

By the definition of projective, $\exists\ \alpha_1$ s.t

$$\partial_N\circ\alpha_1 = \Phi_1-\Psi_1-\alpha_0\circ\partial_N$$

$$\Longleftrightarrow \quad \Phi_1-\Psi_1 = \alpha_0\circ\partial_N + \partial_N\circ\alpha_1$$

Continue this process, we set a chain homotopy, so that $\Phi_* = \Psi_*$. $\Rightarrow$ Φ is unique up to homotopy. ///

Proof of assertion 4.

Given projective resolutions on M' & M'',

$$\to E_1' \to E_0' \to M' \to 0$$

$$\to E_1'' \to E_0'' \to M'' \to 0.$$

Consider $E_1'\oplus E_1'' \to E_0'\oplus E_0'' \to M \to 0$, where maps from $E_n'\oplus E_n''$ to $E_{n-1}'\oplus E_{n-1}''$ are obvious for $n\geq 1$.

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "g"] & M \arrow[r, "h"] & M'' \arrow[r] & 0 \\ 0 \arrow[r] & E_0' \arrow[r] \arrow[u, "f'"] & E_0'\oplus E_0'' \arrow[r] \arrow[u, "?"] & E_0'' \arrow[r] \arrow[u, "f''"'] \arrow[ul, dashed, "\alpha"'] & 0 \end{tikzcd}$$

Homological Algebra
Horse shoe lemma

Define $E_0'\oplus E_0'' \xrightarrow{?} M$ by

$(e_0', e_0'') \longmapsto (g\circ f'(e_0'), \alpha(e_0''))$, where

$(e_0'+e_0'' \longmapsto g\circ f'(e_0')+\alpha(e_0''))$ refer to Rotman P187 ~P188

$h\circ\alpha = f''$ by the definition of projectiveness of E_0''.

Clearly the diagrams are commutative. $\Rightarrow$ $\{E_n'\oplus E_n''\} = E_\cdot(M)$ is projective resolution.