If we have different trivilization ψ_α's,

$$\begin{array}{ccccc} U_\alpha \times \mathbb{C} & \xleftarrow{\psi_\alpha} & L_{U_\alpha} & \xrightarrow{\varphi_\alpha} & U_\alpha \times \mathbb{C} \\ & & \Downarrow & & \\ (x, s_\psi(x)) & \longleftarrow & s & \longmapsto & (x, s_\varphi(x)) \end{array}$$

$\Rightarrow \quad \exists \; h(x) \;$ s.t $\quad s_\varphi(x) = h(x)\, s_\psi(x). \qquad h(x) \in \mathcal{O}^*(U_\alpha).$

$$\varphi_\alpha \circ \psi_\alpha^{-1} (x, s_\psi(x)) = (x, h(x)\, s_\psi(x))$$
$$= h(x)\,(x, s_\psi(x))$$

$$\varphi_\alpha(s) = h(x)\, \psi_\alpha(s)$$

$\Rightarrow \quad g_{\alpha\beta} = g'_{\alpha\beta} \dfrac{h_\alpha}{h_\beta} \qquad h_\alpha, h_\beta$ in $\mathcal{O}^*(U_\alpha)$ & $\mathcal{O}^*(U_\beta)$ respectively

$\Rightarrow \quad \mathrm{ord}_V(s_\varphi) = \mathrm{ord}(s_\psi)$

Thus if D is any divisor such that $[D] = L$, there exists a meromorphic section s of L with $(s) = D$, and for any meromorphic section s of L, $L = [(s)]$.

In particular, we see that L is the line bundle associated to some divisor D on M $\Leftrightarrow$ it has a global meromorphic section not identically zero; it is the line bundle of an effective divisor $\Leftrightarrow$ it has a nontrivial global holomorphic section.

We can also view this correspondence as follows: Given a divisor

$D = \sum a_i V_i$ on M, let $\mathcal{L}(D)$ denote the space of meromorphic functions f on M such that $D + (f) \geq 0$, i.e that are holomorphic on $M - \cup V_i$ with $\mathrm{ord}_{V_i}(f) \geq a_i$.