pf) Since V is irreducible, we need only show that the linear span V' of $\{Y^i v\}$ is fixed under sl_2.
Clearly, $HV' \subset V'$ $\ulcorner H(Yv) = (\lambda+2)Yv \Rightarrow$ again Yv is an eigen-vector for H. Keep going $\Rightarrow HY^k v = (\lambda+2k) Y^k v$. $\lrcorner$ and $YV' \subset V'$. We show $XV' \subset V'$ by an induction:

$Xv = 0$ trivially lies in V', and in general,

$$X\,Y^n v = XY(Y^{n-1}v) = YX(Y^{n-1}v) + HY^{n-1}v;$$

so $\quad X(Y^{n-1}v) \in V' \Rightarrow \quad XY^n v \in V'.$ Q.E.D.

Note that the elements $\{Y^n v\}_n$ that are nonzero are linearly independent, since they are all eigenvectors for H with different eigenvalues.

$\ulcorner$

$$\begin{aligned} a_1 v_1 + a_2 v_2 + a_3 v_3 &= 0 \\ \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2 + \lambda_3 a_3 v_3 &= 0 \\ \lambda_1^2 a_1 v_1 + \lambda_2^2 a_2 v_2 + \lambda_3^2 a_3 v_3 &= 0 \end{aligned}$$

like solving

$$\begin{aligned} a_1 + a_2 + a_3 &= 0 \\ \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 &= 0 \\ \lambda_1^2 a_1 + \lambda_2^2 a_2 + \lambda_3^2 a_3 &= 0 \end{aligned}$$

$$\left\| \begin{pmatrix} 1, & 1, & 1 \\ \lambda_1 & \lambda_2, & \lambda_3 \\ \lambda_1^2, & \lambda_2^2, & \lambda_3^2 \end{pmatrix} \right\| = \begin{vmatrix} 0 & 1 & 0 \\ \lambda_1-\lambda_2 & \lambda_2 & \lambda_3-\lambda_2 \\ \lambda_1^2-\lambda_2^2 & \lambda_2^2 & \lambda_3^2-\lambda_2^2 \end{vmatrix}$$

$$\begin{aligned} &= -(\lambda_1-\lambda_2)(\lambda_3^2-\lambda_2^2) + (\lambda_3-\lambda_2)(\lambda_1^2-\lambda_2^2) \\ &= -(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_3+\lambda_2) + (\lambda_3-\lambda_2)(\lambda_1-\lambda_2)(\lambda_1+\lambda_2) \\ &= -(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_3+\cancel{\lambda_2}-\lambda_1-\cancel{\lambda_2}) \\ &= -(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_3-\lambda_1) \neq 0. \quad \Rightarrow \quad a_1 = a_2 = a_3 = 0. \lrcorner \end{aligned}$$

Thus we have the picture of V: $V = \oplus V_\lambda$, where each V_λ is one-dimensional,

$$H(V_\lambda) = V_\lambda, \qquad X(V_\lambda) = V_{\lambda+2}, \quad Y(V_\lambda) = V_{\lambda-2}.$$