Def: A connection D on a complex vector bundle $E \longrightarrow M$ is a map

$$D : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1(E) \quad \text{satisfying Leibnitz's rule}$$

$$D(f\xi) = df \otimes \xi + f\, D(\xi). \quad \text{for all sections } \xi \in \mathcal{A}^0(E)(U),\ f \in C^\infty(U).$$

A connection is essentially, a way of differentiating sections; for $\xi \in \mathcal{A}^0(E)(U)$, the contraction of $D\xi$ with a tangent vector $v \in T_z(M)$ may be thought of as the derivative of ξ in the direction v. It is, however, only a first-order approximation of differentiation, in as much as mixed partials will in general not be equal.

Let $\{e_1 \cdots e_n\} = e$ frame for E over U. D connection on E,

$\Rightarrow$ $De_i = \sum \theta_{ij} e_j$. $\Rightarrow$ The matrix $\theta = (\theta_{ij})$ of 1-forms is called the connection matrix of D with r. to e.

For $\sigma \in \mathcal{A}^0(E)(U)$, $\sigma = \sum \sigma_i e_i$.

$$\Rightarrow D\sigma = \sum d\sigma_i\, e_i + \sum \sigma_i\, De_i = \sum_j \Big(d\sigma_j + \sum_i \sigma_i \theta_{ij}\Big) e_j.$$

The connection matrix θ at a point $z_0 \in U$ depends on the choice of frame in a nbd of z_0; if $e' = \{e_1' \cdots e_n'\}$ is another frame with $e_i'(z) = \sum g_{ij}(z)\, e_j(z)$.

then

$$\sum \theta'_{ij} e'_j = De_i' = \sum d g_{ij}\, e_j + \sum g_{ik} \theta_{kj}\, e_j \quad \text{so that}$$

$$\sum \theta'_{ij} e'_j = \sum \theta'_{ij}\, g_{jk}(z) e_k(z)$$

$$\theta_{e'} = dg \cdot g^{-1} + g\, \theta_e\, g^{-1}.$$

~~There~~ $\exists$ in general no "natural" connection on E. If M complex & E hermitian, we can make two requirements that dictate a canonical choice of connection.