An ideal sheaf I that locally has a single generator f is locally free of rank one, and hence is of the form
$$I \cong \mathcal{O}(L^*)$$
for some holomorphic line bundle $L^* \longrightarrow M$.

⌈ Consider
$$F : \mathcal{O}|_U \longrightarrow I|_U, \qquad g \longmapsto gf$$

$\Rightarrow$ $R = \{ g \in \mathcal{O}|_U \mid gf = 0 \}$. $\Rightarrow R = 0$ since $f \neq 0$ and $\mathcal{O}_z$ is an integral domain, and $g = 0$. $\Rightarrow \mathcal{O}|_U \cong I|_U$.
$\Rightarrow$ By P687 & P612, note, $I \cong \mathcal{O}(L^*)$, $L^* \to M$ line bundle. ⌋

Denoting by $D = (f)$ the divisor of f, we have previously used the notations $L^* = [-D]$ and $L = [D]$ for this line bundle and its dual.

⌈ f is a meromorphic section for $[D]$. So let f_α be a generator for $I(U_\alpha)$.

$$\begin{array}{ccccc}
U_\beta \times \mathcal{O}(U_\alpha \cap U_\beta) & \xleftarrow{\varphi_\beta} I(U_\alpha\cap U_\beta) = & I(U_\alpha \cap U_\beta) & \xrightarrow{\varphi_\alpha} & U_\alpha \cap U_\beta \times \mathcal{O}(U_\alpha \cap U_\beta) \\
(z, \; 1)\frac{f_\alpha}{f_\beta} & \longleftarrow & f_\beta = \frac{f_\beta}{f_\alpha}\cdot f_\alpha, \; f_\alpha & \longleftarrow & (z, \; 1)
\end{array}$$

$$\varphi_\beta(f_\beta) = 1 \qquad \varphi_\beta(\underset{f_\alpha}{\uparrow}) = \varphi_\beta\left(f_\beta \cdot \frac{f_\alpha}{f_\beta}\right) = \frac{f_\alpha}{f_\beta}\,\varphi_\beta(f_\beta) = \frac{f_\alpha}{f_\beta}$$