Put $z = \frac{1}{z'}$. $\Rightarrow$ $\frac{g(z)}{f(z)} dz = -\frac{1}{z'} \frac{1}{(\alpha_0 z'-1)\cdots(\alpha_n z'-1)} dz'$.

$\Rightarrow$ The differential has a simple pole at $z'=0 (\Leftrightarrow z=\infty)$.

$$2\pi\sqrt{-1}\ \mathrm{Res}_\infty(\varphi) = \int_{B_\epsilon(0)} \frac{-1}{\prod_{i=0}^{n}(\alpha_i z'-1)} \frac{1}{z'} dz'$$

$$= 2\pi\sqrt{-1} \frac{-1}{\prod_{i=0}^{n}(\alpha_i \cdot 0-1)} = 2\pi\sqrt{-1} \frac{(-1)}{(-1)^{n+1}} = 2\pi\sqrt{-1}(-1)^n$$

$\Rightarrow$ $\mathrm{Res}_\infty(\varphi) = (-1)^n$.

By the residue theorem,

$$\sum_i \frac{\alpha_i^n}{\prod_{j\neq i}(\alpha_j-\alpha_i)} = (-1)^n$$

and consequently

$$c_1(\mathbb{P}^n)^n = \sum_i \frac{(-1)^n (n+1)^n \alpha_i^n}{\prod_{j\neq i}(\alpha_j-\alpha_i)} = (n+1)^n;$$

since the nth power of the hyperplane class ω in $\mathbb{P}^n$ is 1, this implies that

$$c_1(\mathbb{P}^n) = (n+1)\omega.$$

By the residue theorem on p222.

$$(-1)^n + \sum_{i=0}^{n} \frac{-\alpha_i^n}{\prod_{j\neq i}(\alpha_j-\alpha_i)} = 0 \Rightarrow \sum \frac{\alpha_i^n}{\prod_{j\neq i}(\alpha_j-\alpha_i)} = (-1)^n.$$