Birational Maps. We say that a rational map $f: M \to N$ is birational if there exists a rational map $g: N \to M$ s.t. $f \circ g$ is the identity as a rational map; two algebraic varieties are said to be birationally isomorphic, or simply birational, if there exists a birational map between them. In particular, a variety is called rational if it is birational to $\mathbb{P}^n$, i.e., if there exist n meromorphic functions on it providing local coordinates almost everywhere.

Γ Let M be birational to $\mathbb{P}^n$.

$\Rightarrow$ $f: M \longrightarrow \mathbb{P}^n$ birational, and $g: \mathbb{P}^n \longrightarrow M$ birational s.t $f \circ g: \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is birational as the identity.

$\Rightarrow$ $\exists$ a subvariety $V \subset \mathbb{P}^n$ s.t

$$\mathbb{P}^n - V \xrightarrow{g} M \xrightarrow{f} \mathbb{P}^n - V \quad \text{is identity holomorphic.}$$

First, g is one to one. for, if $g(p_1) = g(p_2)$, $p_1, p_2 \in \mathbb{P}^n - V$, $f(g(p_1)) = f(g(p_2)) = p_1 = p_2$.

Suppose $f: M - W \longrightarrow \mathbb{P}^n$ is holomorphic.

$\Rightarrow$ $\mathbb{P}^n - V - g^{-1}(W) \xrightarrow{g} M - f^{-1}(V) - W \xrightarrow{f} \mathbb{P}^n - V - g^{-1}(W)$ is well-defined for, $\forall p \in M - f^{-1}(V) - W$, $f(p) \notin V$ and $f(p) \notin g^{-1}(W)$ ($\because$ $g(f(p)) = p \notin W$). $f(p) \in \mathbb{P}^n - V - g^{-1}(W)$

$\Rightarrow$ Consider $g(f(p))$. $\Rightarrow$ $f(g(f(p))) = f(p)$.

$\Rightarrow$ f is onto once we prove $\operatorname{img} \subset M - f^{-1}(V) - W$.

To show this, we have only to prove that $g(\mathbb{P}^n - V - g^{-1}(W))$