Then we have a contradiction. For, $P_1, P_2, P_3, P_4, P_5, P_6, P_7$ fail and $P_1, P_2, P_3, P_4, P_6, P_7, P_8$ OK give $P_1, P_2, P_4, P_5, P_{5+1}, P_7, P_8$ fail, which is absurd.
If P_6, P_7, P_8 are collinear, then we have P_5, P_6, P_7, P_8 collinear by ①. So if not, then choose a conic τ' containing P_1, P_2, P_3, P_4, P_5. Consider $\tau' + L_{67}$, $\tau' + L_{78}$ and $\tau' + L_{68}$. $\Rightarrow$ By the independence of $P_1, P_2, P_3, P_4, P_5, P_7, P_8$'s and the dependence of $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$,

$\tau' + L_{67} \ni P_8$ $\quad \tau' + L_{78} \ni P_6$, $\quad \tau' + L_{68} \ni P_7$

$\Rightarrow \tau' \ni P_6$ and $\tau' \ni P$

If P_6, P_7, P_8 are collinear, then we have P_5, P_6, P_7, P_8 collinear by ①. $\Rightarrow$ done. If not, then choose a conic τ' containing P_1, P_2, P_3, P_4, P_5. Consider $\tau' + L_{67}$, $\tau' + L_{78}$, $\tau' + L_{68}$. $\Rightarrow$ By the dependence of $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8$ and the independence of $P_1, P_2, P_3, P_4, P_i, P_j, P_k$'s $(\neq P_5, P_6, P_7)$

$\Rightarrow \tau' + L_{67} \ni P_8$ (?). $\quad \tau' + L_{78} \ni P_6 \quad \tau' + L_{68} \ni P_7$

$\Rightarrow \tau' \ni P_6, P_7. \Rightarrow$ We can not conclude anything more. !!!

Consider a conic $L_{12} + L_5 = \tau''$ s.t L_5 is a line passing P_5 but P_6, P_7, P_8. $\Rightarrow \tau'' + L_{67} \ni P_8$ (?, by the dependence of $P_1, P_2, P_3, P_4, P_5, P_6, P_7$),

$\tau'' + L_{78} \ni P_6 \Rightarrow P_6 \notin L_{78} \Rightarrow P_6 \in \tau'' = L_{12} + L_5$

$\Rightarrow P_6 \in L_{12}$ done. $\lrcorner$