Date . .

Kodaira Embedding Theorem. A compact complex manifold M is an algebraic variety — i.e., is embeddable in projective space — if and only if it has a closed, positive $(1,1)$-form ω whose cohomology class $[\omega]$ is rational.

pf) If $[\omega] \in H^2(M, \mathbb{Q})$, then for some k, $[k\omega] \in H^2(M, \mathbb{Z})$; in the exact sequence

$$H^1(M, \mathcal{O}^*) \longrightarrow H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathcal{O})$$

$i_*([k\omega]) = 0$, and so there exists a holomorphic line bundle $L \longrightarrow M$ with $c_1(L) = [k\omega]$. The line bundle will then be positive. Q.E.D.

┌ By P163 ~ P164, in the proof of Lefschetz Theorem on $(1,1)$ classes, given $\gamma \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, then $i_*(\gamma) = 0$, where $i_* : H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathcal{O})$.

Thus since $[k\omega] \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})$,

$i_*([k\omega]) = 0 \Rightarrow \exists\ L \rightarrow M$ s.t $c_1(L) = [k\omega]$.

$\Rightarrow$ By P148. Proposition, (L is positive $\Leftrightarrow$ its Chern class may be represented by a positive form in $H^2_{DR}(M)$),

$L \rightarrow M$ is positive line bundle.

Then by Kodaira Embedding Theorem, $\exists\ k_0$ s.t for $k \geq k_0$ the map $i_{L^k} : M \longrightarrow \mathbb{P}^N$ is an embedding.

Conversely, $M \hookrightarrow \mathbb{P}^N$ is an embedding.

$\Rightarrow \exists$ the Fubini-Study metric ω on $\mathbb{P}^N$.

ω is positive $(1,1)$-form on $\mathbb{P}^N$. By P122, $\omega|_M$ is Poincaré dual to the homology class (V) of $V = M \cap H \subset M$.