The correspondence $[\ ]$ has these immediate properties: First, if D and D' are two divisors given by local data $\{f_\alpha\}$ and $\{f'_\alpha\}$, respectively, then.

$D+D'$ is given by $\{f_\alpha \cdot f'_\alpha\}$; it follows that

$$[D+D'] = [D] \otimes [D'].$$

Since $\left\{ g_{\alpha\beta}\, g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \frac{f'_\alpha}{f'_\beta} = \frac{f_\alpha f'_\alpha}{f_\beta f'_\beta} \right\}$ are the transition functions

for $[D] \otimes [D']$ by the arguments on P 133 & P67.

so the map $[\ ] : \mathrm{Div}(M) \longrightarrow \mathrm{Pic}(M)$, $D \longmapsto [D]$ is a homomorphism.

Second, if $D = (f)$ for some meromorphic function f on M, we may take as local data for D over any cover $\{U_\alpha\}$ the functions $f_\alpha = f|U_\alpha$; then $f_\alpha/f_\beta = 1$ and so $[D]$ is trivial.

$\ulcorner$ Since $g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} = 1$, $\quad \dfrac{\amalg U_\alpha \times \mathbb{C}}{(x,z)\sim(x,z)} \cong M \times \mathbb{C}$ $\lrcorner$

Conversely, if D is given by local data $\{f_\alpha\}$ and the line bundle $[D]$ is trivial, then $\exists$ functions $h_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\alpha}{h_\beta};$$

$f = f_\alpha \cdot h_\alpha^{-1} = f_\beta\, h_\beta^{-1}$ is then a global meromorphic function on M with divisor D.

$\ulcorner$ $\dfrac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \dfrac{h_\alpha}{h_\beta} g'_{\alpha\beta}$. where $g'_{\alpha\beta} = 1$. since $[D]$ is

trivial, see P133 (**).

$\Rightarrow$ $f_\alpha h_\beta = f_\beta h_\alpha$ $\Rightarrow$ $f_\alpha h_\alpha^{-1} = f_\beta h_\beta^{-1}$. on $U_\alpha \cap U_\beta$

$\Rightarrow$ f defines a global meromorphic function on M, with