A local base is given by the sets

(5) $$V_N = \{ f \in C^\infty(\Omega) : P_N(f) < \frac{1}{N} \} \quad (N = 1, 2, 3, \cdots).$$

If $\{f_i\}$ is a Cauchy sequence in $C^\infty(\Omega)$ and if N is fixed, then $f_i - f_j \in V_N$ if i and j are sufficiently large.

⌈

$$d(f_i, f_j) = \sum_{\ell=1}^{\infty} \frac{2^{-\ell} P_\ell(f_i - f_j)}{1 + P_\ell(f_i - f_j)} < \delta$$

$$\Rightarrow \quad \frac{2^{-N} P_N(f_i - f_j)}{1 + P_N(f_i - f_j)} < \delta \quad \Rightarrow \quad 2^{-N} P_N(f_i - f_j) < \delta + \delta P_N(f_i - f_j)$$

$$\Rightarrow \quad (2^{-N} - \delta) P_N(f_i - f_j) < \delta \quad \Rightarrow \quad P_N(f_i - f_j) < \frac{\delta}{2^{-N} - \delta}$$

Since N is fixed, choose δ sufficienly small so that $\frac{\delta}{2^{-N} - \delta} < \frac{1}{N}$. $\Rightarrow$ For such δ, $\exists$ N_δ s.t if $i, j \geq N_\delta$, then $P_N(f_i - f_j) < \frac{1}{N}$ if $d(f_i, f_j) < \delta$.

⌋

Thus $| D^\alpha f_i - D^\alpha f_j | < \frac{1}{N}$ on K_N, if $|\alpha| \leq N$. It follows that each $D^\alpha f_i$ converges (uniformly on compact subsets of Ω) to a function g_α. In particular, $f_i(x) \longrightarrow g_0(x)$. It is now evident that $g_0 \in C^\infty(\Omega)$, that $g_\alpha = D^\alpha g_0$, and $f_i \rightarrow g$ in the topology of $C^\infty(\Omega)$.

⌈ We have only to prove the following: