$\tilde{D}_3 = \left(\frac{\pi^* f_3}{z_i^{m_1}} = 0 \right)$ $\qquad \pi^* f_1 + \pi^* f_2 = \pi^* f_3$ and

$$-\frac{\pi^* f_1}{z_i^{m_1}} + \frac{\pi^* f_3}{z_i^{m_1}} = \frac{\pi^* f_2}{z_i^{m_1}} .$$

This is absurd, because

$$\tilde{D}_2 = \left(\frac{\pi^* f_2}{z_i^{m_2}} = 0 \right)$$

, and $\quad \frac{\pi^* f_1}{z_i^{m_1}} - \frac{\pi^* f_3}{z_i^{m_1}} \neq -\frac{\pi^* f_2}{z_i^{m_2}}$, $\Leftrightarrow \frac{\pi^* f_1}{z_i^{m_1}} + \frac{\pi^* f_2}{z_i^{m_2}} \neq \frac{\pi^* f_3}{z_i^{m_1}}$

Suppose D has multiplicity of m_0 at
$\Rightarrow$ $(D)+(D_\lambda)$ has multiplicity of m_0 at p, $\mathrm{mult}_p(D_\lambda) \geq m_0$.
It is easy to see by looking at the following:
Let $D = (f)$. $\quad f = \sum_{m \geq m_0} f_m$
$\quad D_\lambda = (f_\lambda)$, $\quad f_\lambda = \sum_{m \geq m_0} (f_\lambda)_m$.

$\Rightarrow \quad D' = (f + f_\lambda) \quad \Rightarrow \quad f + f_\lambda = \sum_{m \geq m_0} f_m + (f_\lambda)_m$.
$\Rightarrow$ The generic curve D_λ has multiplicity $m_0 = \min\{\mathrm{mult}_p(D_\lambda)\}_\lambda$.

$\{\pi^* D_\lambda - m_0 E\}$ is a linear system.

$$\pi^* D_\lambda - m_0 E = \pi^* D_\lambda - \mathrm{mult}_p(D_\lambda) + \mathrm{mult}_p(D_\lambda) - m_0 E$$

$$= \left(\frac{\pi^* f_\lambda}{z_i^{m_\lambda}} \times \frac{z_i^{m_\lambda}}{z_i^{m_0}} = 0 \right) = \left(\frac{\pi^* f_\lambda}{z_i^{m_0}} \right)$$

$$\Rightarrow \quad \frac{\pi^* f_\lambda}{z_i^{m_0}} + \frac{\pi^* f_{\lambda'}}{z_i^{m_0}} = \frac{\pi^* f_{\lambda''}}{z_i^{m_0}} , \text{ where } f_\lambda + f_{\lambda'} = f_{\lambda''} .$$

$\square$

We see from all the above that the blow-up $\tilde{M}$ of a surface M is very closely related to M. An important question to ask, then, is the converse: