A set P_0 of eight points imposes independent conditions on $|\mathcal{O}_{\mathbb{P}^2}(3)|$, unless five are on a line or all eight are on a conic.

Proof. If we assume that P_0 fails to impose independent conditions, then $\dim|\mathcal{J}_{P_0}(3)| \geq 2$ and, as before, we conclude that any two cubics from this linear system have a common component C_0.

Γ By the formula on P713, $\dim|\mathcal{J}_{P_0}(3)| = \frac{3(3+3)}{2} - 8 + h^1(\mathcal{J}_{P_0}(3))$

$= 1 + h^1(\mathcal{J}_{P_0}(3)) = 1 + \omega \geq 2$, since $\omega \geq 1$ ($\because$ P_0 fails to impose independent condition ($\Leftrightarrow \omega \neq 0$)).

Suppose $|\mathcal{J}_{P_0}(3)|$ does not satisfy case 1 on P714 i.e $|\mathcal{J}_{P_0}(3)|$ has no fixed curve.

$\Rightarrow$ $|\mathcal{J}_{P_0}(3)|$ must satisfy case 2. $\Rightarrow$ By the Reciprocity Formula I,

$$\dim|\mathcal{J}_{P_0}(3)| = 1 + h^0(\mathcal{J}_P(0))$$

$h^0(\mathcal{J}_P(0)) = \dim H^0(\mathbb{P}^2, \mathcal{J}_P(0)) =$ dim of the space of holomorphic function on $\mathbb{P}^2$ with nonempty zeros, since $3^2 - 8 = 1 = \#P$.

As we know $H^0(\mathbb{P}^2, \mathcal{O}) = \mathbb{C}$, $h^0(\mathcal{J}_P(0)) = 0$.

$\Rightarrow$ Contradiction to $\dim|\mathcal{J}_{P_0}(3)| \geq 2$.

Thus $|\mathcal{J}_{P_0}(3)|$ must have a fixed curve of deg less than 3. $\lrcorner$

If C_0 is a conic, then since $\dim|\mathcal{J}_{P_0}(3)| \geq 2$, all