Here, of course, we are writing $H^2(M, \mathbb{Z})$ for its image under the natural inclusion in $H^2(M, \mathbb{R})$.

pf). Consider again the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}^* \longrightarrow 0.$$

and the associated cohomology sequence

$$H^1(M, \mathcal{O}^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \xrightarrow{i_*} H^2(M, \mathcal{O}) \cong H^{0,2}_{\bar\partial}(M).$$

We claim that the map i_* is given by, first, mapping $H^2(M, \mathbb{Z}) \longrightarrow H^2(M, \mathbb{C})$ and then projecting onto the $(0,2)$-factor of $H^2(M, \mathbb{C})$ in the Hodge decomposition; i.e. that the diagram

$$\begin{tikzcd}
H^2(M, \mathbb{Z}) \arrow[r, "i_*"] \arrow[d] & H^2(M, \mathcal{O}) \arrow[dd, "\cong \text{ Dolbeault}"] \\
H^2(M, \mathbb{C}) \arrow[d, "\text{de Rham}"', "\cong"] & \\
H^2_{DR}(M, \mathbb{C}) \arrow[r, "\pi^{0,2}"] & H^{0,2}_{\bar\partial}(M)
\end{tikzcd}$$

Commutes. (The map $\pi^{0,2}$ is defined on the form level, since for $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2} \in Z^2_d(M)$, $\bar\partial \omega^{0,2} = (d\omega)^{0,3} = 0$).

Γ $(d\omega)^{0,3} = (d\omega^{2,0} + d\omega^{1,1} + d\omega^{0,2})^{0,3}$
$= (d\omega^{0,2})^{0,3} = ((\partial + \bar\partial)\omega^{0,2})^{0,3} = \bar\partial \omega^{0,2}$ $\lrcorner$