$$= H_{r-k}(E.(f)) = \begin{cases} 0 & \text{if } r-k>0 \\ \mathcal{O}/I & \text{if } r-k=0 \end{cases}$$

To prove the transformation formula (**), we shall define mappings

$$\begin{tikzcd}
E.(f): 0 \arrow[r] & E_r \arrow[r] & E_{r-1} \arrow[r] & \cdots \arrow[r] & E_k \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & E_0 \arrow[r] & \mathcal{O}/I \arrow[r] & 0 \\
E.(f'): 0 \arrow[r] & E_r' \arrow[r] \arrow[u, "A_r"] & E_{r-1}' \arrow[r] \arrow[u, "A_{r-1}"] & \cdots \arrow[r] & E_k' \arrow[r] \arrow[u, "A_k"] & \cdots \arrow[r] & E_1' \arrow[r] \arrow[u, "A_1"] & E_0' \arrow[r] \arrow[u, "A_0"] & \mathcal{O}/I' \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

between the Koszul complexes as follows: The map $\mathcal{O}/I' \to \mathcal{O}/I$ is the natural map induced by the inclusion $I' \subset I$, and A_0 is the identity under the identifications $E_0 \cong \mathcal{O} \cong E_0'$. $A_1: E_1' \to E_1$ is defined by

$$A_1(e_i') = \sum_j a_{ij} e_j,$$

so that $\partial A_1(e_i') = \sum a_{ij} f_j = f_i' = A_0(\partial e_i')$.

Pf $\quad A_1 : E_1' = \mathcal{O} \otimes_{\mathbb{C}} \wedge \mathbb{C}^{r-1} \longrightarrow E_1 = \mathcal{O} \otimes_{\mathbb{C}} \wedge \mathbb{C}^{r-1}$

$$e_i' \longmapsto \sum_j a_{ij} e_j$$

$$\partial A_1(e_i') = \sum_j a_{ij} \partial e_j = \sum_j a_{ij} f_j = f_i' = A_0(\partial e_i') = \partial e_i' = f_i'$$

The remaining maps $A_k: E_k' \longrightarrow E_k$ are the kth exterior powers of A_1. The diagram is then commutative. Under the identifications