Review of Hodge Theory for $A^*(M)$

(i) Fix $\omega \in \mathcal{H}_0^{p,q}(M)$, consider $\phi(\eta) = \langle \omega, \eta \rangle$, $\eta \in A^{p,q}(M)$.

$\Rightarrow$ $|\phi(\eta)| = |\langle \omega, \eta \rangle| \le \|\omega\|_0 \|\eta\|_0 \le \|\omega\|_0 \|\eta\|_1$
$\le M \|\omega\|_0 \mathcal{D}(\eta)$, for some constant $M > 0$.
since the Dirichlet norm $\mathcal{D}$ is equivalent to the Sobolev 1-norm. (This follows from the Garding's inequality.)

$\Rightarrow$ ϕ can be extended to $\mathcal{H}_1^{p,q}(M)$, since ϕ is uniformly continuous.

Since ϕ is bounded linear functional on $\mathcal{H}_1^{p,q}(M)$ with the inner product called Dirichlet. and $\mathcal{H}_1^{p,q}(M)$ is complete, by P85. Th 4.12. Rudin R&C.A,

$\exists ! \psi \in \mathcal{H}_1^{p,q}(M)$ s.t $\phi(\eta) = \langle\langle \psi, \eta \rangle\rangle$. for all $\eta \in \mathcal{H}_1^{p,q}(M)$.
where $\langle\langle \psi, \eta \rangle\rangle = \mathcal{D}(\psi, \eta) = \langle \psi, (I+\Delta)\eta \rangle$.

$\Rightarrow$ $\langle \omega, \eta \rangle = \langle \psi, (I+\Delta)\eta \rangle$

Define $T : \mathcal{H}_0^{p,q}(M) \longrightarrow \mathcal{H}_1^{p,q}(M)$ by
$T(\omega) = \psi$.

$\Rightarrow$ According to P95, T is bounded, & adjoint.

$T : \mathcal{H}_0^{p,q}(M) \xrightarrow{T} \mathcal{H}_1^{p,q}(M) \hookrightarrow \mathcal{H}_0^{p,q}(M)$ is compact.

(ii). By the spectral theorem for compact, self-adjoint operator, there is a Hilbert-space decomposition.