Using the resolution of singularities theorem*, the second isomorphism

$$\mathbb{H}^*(M, \Omega^*(*D)) \cong H^*(M-D, \mathbb{C})$$

holds with no assumptions on the singularities of D.

Suppose now that the line bundle $[D] \longrightarrow M$ is positive. By Theorem B,

$$H^q(M, \Omega^p(kD)) = 0 \quad \text{for } q>0, \ k \geq k_0.$$

┌ By Theorem B on P159, $H^q(M, \mathcal{O}([D]^k \otimes \Lambda^p T^{*'}M) = 0$ for $q>0$, $k \geq k_0$.

$\Rightarrow \ \mathcal{O}([D]^k \otimes \Lambda^p T^{*'}M) = \Omega^p(kD)$. ┘

If we set $U = M-D$ and denote by

$$H^*_{DR}(U, alg)$$

the cohomology of the complex of meromorphic forms that are holomorphic in U and have poles on D, then by the degeneration of the second spectral sequence of hypercohomology we obtain

Grothendieck's Algebraic de Rham Theorem

$$H^*_{DR}(U, alg) \cong H^*(U, \mathbb{C}).$$

┌ Claim: $H^q(M, \Omega^p(*D)) = 0$ for $q>0$.

pf). $H^q(M, \Omega^p(*D)) \ni \sigma$.

$\Rightarrow$ For some open covering $\underline{U}$ of M,

$$\begin{array}{ccc} C^p(\underline{U}, \Omega^p(*D)) & \xrightarrow{\delta} & C^{p+1}(\underline{U}, \Omega^p(*D)) \\ \sigma & \longmapsto & 0. \end{array}$$