The second expression shows that $\rho(x)$ is smooth and

$$\Delta \rho(x) = \pm C_n \int_{u \in \mathbb{R}^n} \frac{\Delta \eta(x-u)\, du}{\|u\|^{n-2}}.$$

$\ulcorner$ By p 370, $\|u\|^{-(n-2)}$ is integrable. $\Rightarrow$ $\rho(x) = \pm C_n \int_{u \in \mathbb{R}^n} \frac{\eta(x-u}{\|u\|^{n-2}}$

$) du$ is C^∞. $\lrcorner$

We will prove that this integral is $\eta(x)$. By translating x to the origin, what must be verified is the Poisson formula

$$\eta(0) = C_n \int_{\mathbb{R}^n} \frac{\Delta \eta(x)}{\|x\|^{n-2}} dx$$

$\ulcorner$ $\pm C_n \int_{u \in \mathbb{R}^n} \frac{\Delta \eta(x-u)}{\|u\|^{n-2}} du = \eta(x)$

Let $x=0$. $\Rightarrow$ $\eta(0) = \pm C_n \int_{u \in \mathbb{R}^n} \frac{\Delta \eta(-u)}{\|u\|^{n-2}} du$

$= C_n \int_{u \in \mathbb{R}^n} \frac{\Delta \eta(x)}{\|x\|^{n-2}} dx$ (let $x = -u$).

$\Rightarrow$ If we prove the Poisson formula, i.e.,

$$\eta(0) = C_n \int_{u \in \mathbb{R}^n} \frac{\Delta \eta(x)}{\|x\|^{n-2}} dx,$$

consider $\eta(x) = \varphi(x+u)$.