The Quadric Surface

We now consider a smooth surface S of degree 2 in $\mathbb{P}^3$. Such a surface is given as the locus of

$$(x \cdot Qx) = \langle x, Qx \rangle = \sum q_{ij} x_i x_j = 0$$

for $Q = (q_{ij})$ a symmetric matrix; since

$$\frac{\partial}{\partial x_i}(x \cdot Qx) = 2 \sum_j q_{ij} x_j ,$$

we see that S is smooth exactly when the matrix Q is nonsingular.

For example, $ax^2 + bxy + cy^2 = ax^2 + \frac{b}{2} xy + \frac{b}{2} yx + cy^2 = \sum_{i,j=1}^{2} q_{ij} x_i x_j$ $\quad x_1 = x \quad x_2 = y$

$q_{11} = a \quad q_{22} = c \quad q_{12} = q_{21} = \frac{b}{2}$.

$$\left(\frac{\partial (x \cdot Qx)}{\partial x_1}, \ \dots \ \frac{\partial (x \cdot Qx)}{\partial x_n} \right) = 2\left(\sum_j q_{1j} x_j, \ \sum_j q_{2j} x_j, \cdots \sum_j q_{nj} x_j \right)$$

$$= 2 \begin{pmatrix} q_{11}, & q_{12}, & \cdots & q_{1n} \\ \vdots & & & \\ q_{n1}, & q_{n2} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \neq 0 \iff S \text{ is smooth.}$$

for all $(x_1 \cdots x_4) \neq 0$.

$\Updownarrow$

Q is nonsingular.

All nondegenerate symmetric quadratic forms on $\mathbb{C}^4$ are