C (and hence nonzero on C), and set

$$z_1 = \frac{\zeta_1}{\zeta_0}, \quad z_2 = \frac{\zeta_2}{\zeta_0}.$$

⌈ We know that $|L+kC|$ has no base points for $k \le m$.
$\Rightarrow$ $|L'|$ has no base points since $L' = L + mC$.
$\Rightarrow$ $\exists$ $p \in C$ and $\sigma \in H^0(M, \mathcal{O}(L'))$ s.t $\sigma(p) \neq 0$.
$\Rightarrow$ By the argument above, $\sigma \neq 0$ on C, i.e., σ is nonvanishing on C. Let $\sigma = \zeta_0$. ⌋

Let $U_1 = C - \{p_{1+1}\}$, $U_2 = C - \{p_{2-1}\}$. Then in some open set $\tilde{U}_1 \subset M$ containing U_1, z_1/z_2 is holomorphic; in fact, for $p \in U_1$ we have $d(z_1/z_2) \neq 0$ on $T'_p(C) \subset T'_p(M)$ and $dz_2 \neq 0$ on $T'_p(M)/T'_p(C)$, so that if we choose $\tilde{U}_1$ sufficiently small, we may take z_2, z_1/z_2 local coordinates on $\tilde{U}_1$.

⌈ $z_1 = \frac{\zeta_1}{\zeta_0}$, $z_2 = \frac{\zeta_2}{\zeta_0}$ are meromorphic functions on M
$\zeta_1(p_1) = 0 \Rightarrow z_1(p_1) = 0$. $\zeta_2(p_2) = 0 \Rightarrow z_2(p_2) = 0$.
$\frac{z_1}{z_2} = \frac{\zeta_1}{\zeta_2}$ meromorphic function on M and $\frac{z_1}{z_2}$ has no singularity on $U_1 = C - \{p_2\}$
$\Rightarrow$ $\frac{z_1}{z_2}$ is holomorphic on $M - \{\zeta_2 \otimes \tau^{-1} = 0\}$ and $M - \{\zeta_2 \otimes \tau^{-1} = 0\}$ contains U_1 $\Rightarrow$ $\exists$ an open set $\tilde{U}_1$ s.t $\tilde{U}_1 \supset U_1$ and z_1/z_2 is holomorphic in $\tilde{U}_1$.
Missing point: $z_1 = \frac{\zeta_1 \otimes \tau^{-1}}{\zeta_0 \otimes \tau^{-1}}$ vanishes at only p_1 on C
Similarly for z_2.