$t x\, {}^t A E A x = {}^t(Ax) E (Ax), \quad E = (Q(e_i, e_j))$.
If we let $(Q(v_i, v_j)) = V = I$ as we get above,
$$x_1^2 + x_2^2 = {}^t x\, {}^t A E A x.$$
$\Rightarrow$ Thus, by changing basis, we have the quadratic form $x_1^2 + x_2^2$, instead $Q(x_i e_i, x_j e_j) = x_i x_j Q(e_i, e_j) = {}^t x E x$.
$\Rightarrow$ Any two smooth quadric surfaces in $\mathbb{P}^3$ are projectively isomorphic by $A : \mathbb{C}^4 \longrightarrow \mathbb{C}^4$. ⏎

Consider in particular the Segré map
$$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$$
given by
$$([s_0, s_1], [t_0, t_1]) \longmapsto [s_0 t_0, s_0 t_1, s_1 t_0, s_1 t_1].$$
┌ See p192 ⏎

σ is clearly an embedding, and the image of σ is contained in – hence equal to – the smooth quadric
$$S_0 = (X_1 X_4 - X_2 X_3 = 0).$$
Thus any quadric surface $S \subset \mathbb{P}^3$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

┌ Assume $s_0 \neq 0 \neq t_0$ since it is symmetric.
$\Rightarrow \sigma : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$
$$(x, y) \longmapsto (y, x, xy), \quad x = \frac{s_1}{s_0},\ y = \frac{t_1}{t_0}.$$
$\Rightarrow$ Clearly σ is embedding. $\Rightarrow$ $\operatorname{im}\sigma$ is an algebraic smooth surface in $\mathbb{P}^3$, i.e., a quadric surface.
Consider $S_0 = (X_1 X_4 - X_2 X_3 = 0)$. $s_0 t_0 s_1 t_1 - s_0 t_1 s_1 t_0 = 0$
$\Rightarrow \operatorname{im}\sigma \subset S_0$. $\Rightarrow \operatorname{im}\sigma = S_0$, since S_0 and σ are irred-