$\begin{pmatrix} x_{1,\ell+1} \\ \vdots \\ x_{k,\ell+1} \end{pmatrix}$. This implies that $\det \begin{pmatrix} x_{11} \cdots \widehat{x_{1i}} & x_{1k} , x_{1,\ell+1} \\ x_{21} & x_{2,\ell+1} \\ \vdots & \vdots \\ x_{k1} & x_{kk} , x_{k,\ell+1} \end{pmatrix}$

is not zero. This contradicts to the fact that all det except $\det\begin{pmatrix} x_{11} & * \\ * & x_{kk} \end{pmatrix}$ are zero.

Thus we can conclude that $\begin{pmatrix} x_{1,\ell+1} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{k,\ell+1} & & x_{kn} \end{pmatrix}$ is zero matrix, which means that $v_1 \cdots v_k \in W$.

Recall the contraction operator

$$i(v^*) : \Lambda^k V \longrightarrow \Lambda^{k-1} V$$

defined for $v^* \in V^*$ by

$$\langle i(v^*)\Lambda, \Xi \rangle = \langle \Lambda, v^* \wedge \Xi \rangle$$

for all $\Xi \in (\Lambda^{k-1}V)^* \cong \Lambda^{k-1}V^*$.

☞ $i(v^*) \in \mathrm{Hom}(\Lambda^k V, \Lambda^{k-1}V) \Rightarrow \Lambda \in \Lambda^k V$.
$\Rightarrow i(v^*)\Lambda = (i(v^*))(\Lambda) \in \Lambda^{k-1}V$.

We associate to Λ the linear spaces

$$\Lambda^\perp = \{ v^* \in V^* : i(v^*)\Lambda = 0 \} \subset V^*$$

and

$$W = \mathrm{Ann}(\Lambda^\perp) \subset V.$$

☞ See p 200. $W = \mathrm{Ann}(\Lambda^\perp) = \{ v \in V \mid$