「 $$\|u_\epsilon - u\|_0^2 = \int_{\mathbb{R}^n} |u_\epsilon - u|^2 dx < \delta,$$ for sufficiently

small ϵ, since $u_\epsilon \to u$ is uniformly convergent and u_ϵ & u have the compact support in some closed ball.」

If we can prove that the Sobolev norms

$$\|u_\epsilon\|_{s+1}$$

is uniformly bounded for $0 < \epsilon < \epsilon_0$, then, taking a sequence $\epsilon_k \downarrow 0$, a subsequence of u_{ϵ_k} will converge weakly to an element u' of $\mathcal{H}_{s+1}$, and u' must be equal to u.

「 First of all, we need to know what $\mathcal{H}_s(\mathbb{R}^n)$ is. By p55 ~ p59, Yosida, $\mathcal{H}_s(\mathbb{R}^n)$ is the completion of $C_c^s(\mathbb{R}^n)$ with respect to the norm given by.

$$\|\varphi\|_s^2 = \sum_{|\alpha| \le s} \int_{\mathbb{R}^n} |D^\alpha s(x)|^2 dx$$

$\Rightarrow$ Since $C_c^\infty(\mathbb{R}^n)$ is dense in $C_c^s(\mathbb{R}^n)$ w.r.s. $\| \ \|_s$, $\mathcal{H}_s(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$. Given any $f \in C_c^s(\mathbb{R}^n)$, consider f_ϵ. $\Rightarrow$ $f_\epsilon \in C_c^\infty(\mathbb{R}^n)$ and $D^\alpha f_\epsilon \to D^\alpha f$ uniformly by using the argument on p395.

Consider $f_\epsilon(\varphi) = \langle u_\epsilon, \varphi \rangle$. $\varphi \in C_c^\infty(\mathbb{R}^n)$.

$\Rightarrow$ f_ϵ can be extended to $\mathcal{H}_0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, since