Now, since η is d-, ∂-, or $\bar\partial$-exact, its harmonic projection under any of the above Laplacians is zero.

⌈ If η is $\bar\partial$-exact, $\eta = \bar\partial \xi$.

By Hodge theorem, η can be expressed as

$\mathcal{H}(\eta) + \bar\partial(\bar\partial^* G\eta) + \bar\partial^*(\bar\partial G\eta)$ uniquely.

$\Rightarrow$ $\bar\partial(\bar\partial^* G\eta) = \bar\partial\xi$. $\mathcal{H}(\eta) = 0$ $\bar\partial^*(\bar\partial G\eta) = 0$.
$\Rightarrow$ The harmonic projection is zero. ⌋

⌈. If η is d or ∂-exact, its harmonic projection under any of the above Laplacians is zero, since by Hodge theory, the decomposition is unique. ⌋

I think that the lemma is wrong.

η can only be d-exact. $\Rightarrow$ $\eta = d\xi$ $\Rightarrow$ $d\eta = 0$
$\Rightarrow$ $\partial\eta = 0$ $\bar\partial\eta = 0$.

By the Hodge decomposition for $\bar\partial$,

$\eta = \bar\partial\bar\partial^* G_{\bar\partial}\eta$, since $\eta = \mathcal{H}(\eta) + \bar\partial\bar\partial^* G_{\bar\partial}\eta$

$+ \bar\partial^*\bar\partial G_{\bar\partial}\eta$. ($\bar\partial^*\bar\partial G_{\bar\partial}\eta = \bar\partial^* G_{\bar\partial}\bar\partial\eta = \bar\partial^* G_{\bar\partial} 0 = 0$.).

But $\bar\partial^* G_{\bar\partial}\eta$ has pure type (p. 9-1) and so

$\partial(\bar\partial^* G_{\bar\partial}\eta) = \pm\bar\partial^* G_{\bar\partial}(\partial\eta) = 0$.

⌈ I think this is wrong, if η is $\bar\partial$-exact.
We can not say that $\partial\eta = 0$.