exists a rank-two holomorphic vector bundle $E \to S$ with given first Chern class and section $s \in H^0(S, \mathcal{O}(E))$ whose divisor (s) is Z ideal theoretically?

To answer this question, suppose first that $E \to S$ is the rank-two bundle and $s \in H^0(S, \mathcal{O}(E))$ the holomorphic section with divisor $(s) = Z$ that we are trying to construct. We may consider s as a sheaf mapping $\mathcal{E}^* \xrightarrow{s} \mathcal{O}$, $\mathcal{E}^* = \mathcal{O}(E^*)$, and what we are asking for is a short exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{L} \xrightarrow{i} \mathcal{E}^* \xrightarrow{s} \mathcal{I} \longrightarrow 0$$

where $\mathcal{L}$ is locally free of rank one.

$\ulcorner$

$$\begin{array}{ccc} \mathcal{E}^*(U) & \xrightarrow{s} & \mathcal{O}(U) \\ \| & & \\ \mathrm{Hom}(E, \mathcal{O})(U) & & \Downarrow \\ \Downarrow \phi & \longmapsto & \phi(s) \end{array}$$

$\Rightarrow$ s may be considered as a sheaf mapping $\lrcorner$

Proof. Locally, $\mathcal{E} \cong \mathcal{O} \oplus \mathcal{O}$ and $s = (f_1, f_2)$. The map $\mathcal{E}^* \xrightarrow{s} \mathcal{I}$ is given by

$$(g_1, g_2) \longmapsto f_1 g_1 + f_2 g_2,$$

and by comparison with the Koszul complex — of which this is the first step — $\mathcal{L}$ is locally isom-