${}^t g_{\beta\alpha} v \quad \Rightarrow$ By the def. of f. $w = {}^t g_{\beta\alpha} v$

Thus $\psi_\alpha \circ \psi_\beta^{-1}(v) = f \circ \varphi_\alpha^* \circ \varphi_\beta^{*-1} \circ f^{-1}(v)$

$= {}^t g_{\beta\alpha}(v)$

$\Rightarrow \quad \psi_\alpha \circ \psi_\beta^{-1} = {}^t(g_{\alpha\beta}^{-1})$ since $g_{\beta\alpha} g_{\alpha\beta} = id$

$= g_{\alpha\beta} g_{\beta\alpha}$.

Now we want to prove the following "isomorphism"

$$\begin{tikzcd} Q \arrow[r, "f"] \arrow[d] & S^* = \{ (*\Lambda, l) \in G(n-k, n^*) \times \mathbb{C}^{n*} \mid l \in *\Lambda \}^* \arrow[d] \\ G(k, n) \arrow[r] & G(n-k, n^*) \end{tikzcd}$$

defined by $f(\Lambda, w + \Lambda)(*\Lambda, l)$ ($\in S$)

$= l(w)$, $\quad w \in \mathbb{C}^n$.

Clearly f is well-defined, since $l(v) = 0$ for all $v \in \Lambda$.

If $f(\Lambda, w+\Lambda)(*\Lambda, l) = 0$ for all $l \in *\Lambda$,

$l(w) = 0$ for all $l \in *\Lambda$. $\Rightarrow w \in \Lambda$

$\Rightarrow w + \Lambda = \Lambda$. $\Rightarrow$ f is injective

$\Rightarrow$ f is isomorphic on each fiber.

$\Rightarrow Q \cong S^* \Rightarrow Q$ is the pull-back of S^* through $*$, by p 26 lemma 3.1. Milnor.

In other words, S^* = pull-back of Q through $*$.