We want to make one more remark before going on to consider line bundles. On a Riemann surface M, any point is an irreducible analytic hypersurface, and so clearly $\mathrm{Div}(M)$ is always large. This is, in a sense, misleading: a complex manifold M of dimension greater than one need not have any nonzero divisors on it at all. If, however, M is embedded in Projective space $\mathbb{P}^n$, the intersection of M with hyperplanes in $\mathbb{P}^n$ generate a large number of divisors. In fact, among all compact complex manifolds those which are embeddable in projective space can be characterized by having "sufficiently many" divisors, in a sense that we shall make precise in later sections.

Line Bundles.

All line bundles discussed in this section are taken to be holomorphic. Recall that for any holomorphic line bundle $L \xrightarrow{\pi} M$ on the complex manifold M, we can find an open cover $\{U_\alpha\}$ of M and trivializations

$$\varphi_\alpha : \quad L_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C} \quad \text{of} \quad L_{U_\alpha} = \pi^{-1}(U_\alpha).$$

We define the transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$ for L relative to the trivializations $\{\varphi_\alpha\}$ by

$$g_{\alpha\beta}(z) = (\varphi_\alpha \circ \varphi_\beta^{-1})|_{L_z} \in \mathbb{C}^*.$$

The functions $g_{\alpha\beta}$ are clearly holomorphic, nonvanishing, and satisfy

$$(*) \qquad \begin{cases} g_{\alpha\beta}\, g_{\beta\alpha} = 1 \\ g_{\alpha\beta}\, g_{\beta\gamma} \cdot g_{\gamma\alpha} = 1; \end{cases}$$

conversely, given a collection of functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ satisfying these identities, we can construct a line bundle L with transition functions $\{g_{\alpha\beta}\}$ by taking the union of