Let R be the module of relations defined by

$$0 \longrightarrow R \longrightarrow E \longrightarrow M \longrightarrow 0.$$

By the exact sequence of $\operatorname{Tor}_k^{\mathcal{O}}(\mathbb{C}, \cdot)$,

$$\begin{array}{ccccccccc}
\operatorname{Tor}_1^{\mathcal{O}}(\mathbb{C}, M) & \longrightarrow & \mathbb{C}\otimes_{\mathcal{O}} R & \longrightarrow & \mathbb{C}\otimes_{\mathcal{O}} E & \longrightarrow & \mathbb{C}\otimes_{\mathcal{O}} M & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \longrightarrow & R_0 & \longrightarrow & E_0 & \xrightarrow{\sim} & M_0 & &
\end{array}$$

$$\Rightarrow R_0 = 0$$
$$\Rightarrow R = 0.$$

by Nakayama again. Thus $E \cong M$ and the lemma is proved.

Pf

$$\begin{array}{ccccccccc}
\operatorname{Tor}_1^{\mathcal{O}}(\mathbb{C}, M) & \longrightarrow & \operatorname{Tor}_0^{\mathcal{O}}(\mathbb{C}, R) & \longrightarrow & \operatorname{Tor}_0^{\mathcal{O}}(\mathbb{C}, E) & \longrightarrow & \operatorname{Tor}_0^{\mathcal{O}}(\mathbb{C}, M) & \longrightarrow & 0 \\
\| & & \| & & \| & & \| & & \\
0 & & \mathbb{C}\otimes R & & \mathbb{C}\otimes E & & \mathbb{C}\otimes M & & \\
 & & \| & & \| & & \| & & \\
 & \longrightarrow & R_0 & \longrightarrow & E_0 & \xrightarrow{\sim} & M_0 & &
\end{array}$$

$$\Rightarrow R_0 = 0 \Rightarrow R/mR = R_0 = 0$$

$\Rightarrow$ $R = mR$ $\Rightarrow$ By Nakayama lemma, $R = 0$.

$\Rightarrow$ $E \cong M$ $\Rightarrow$ M is free.

If M is free, $0 \longrightarrow \overset{=F_0}{M} \longrightarrow M \longrightarrow 0$.

$\Rightarrow$ $0 \to \underset{=F_1}{0} \to \mathbb{C}\otimes \underset{=F_0}{M} \xrightarrow{id} \mathbb{C}\otimes M \longrightarrow 0$

$\Rightarrow$ $\operatorname{Tor}_1^{\mathcal{O}}(\mathbb{C}, M) = 0$.

Now we can prove the syzygy theorem. For $0 \leq k \leq n$, we define R_k by