Now set

$$Z_r = \sigma_{n-k,\dots,n-k,\,n-k-1,\dots,n-k-1}(V) = \{\Lambda ;\ V_{k-r} \subset \Lambda \subset V_{k+1}\}.$$

$\sigma_{n-k,\dots,n-k,\dots}(V) \ni \Lambda \Rightarrow \dim(V_{n-k+k-r+1-a_{k-r+1}} \cap \Lambda) \geq k-r+1 \Rightarrow \dim(V_{n-k+k-r+1-(n-k-1)} \cap \Lambda) = \dim(V_{k-r+2} \cap \Lambda) \geq k-r+1.$ → Wrong start ←

Since $a_{k-r} = n-k$, $\dim(\Lambda \cap V_{n-k+k-r-(n-k)}) \geq k-r$
$\Rightarrow \dim(\Lambda \cap V_{k-r}) \geq k-r \Rightarrow \Lambda \supset V_{k-r}$.
Since $a_k = n-k-1$, $\dim(V_{n-k+k-(n-k-1)} \cap \Lambda) \geq k$.
$\Rightarrow \dim(V_{k+1} \cap \Lambda) \geq k$, $\Rightarrow \Lambda \subset V_{k+1}$. since $\dim \Lambda = k$. ·⊛
$\Rightarrow V_{k-r} \subset \Lambda \subset V_{k+1}$.

It remains to check that

$$c_r(S)(Z_r) = (-1)^r.$$

To see this, let

$$Z_k = \sigma_{n-k-1,\dots,n-k-1}(V) = \{\Lambda : \Lambda \subset V_{k+1}\} \subset G(k,n).$$

By ⊛ above, if $\Lambda \in Z_k$, then $\Lambda \subset V_{k+1}$.
Conversely, if $\Lambda \subset V_{k+1}$, $\dim(\Lambda \cap V_{n-k+k-r-a_{k-r}}) \overset{?}{\geq} k-r$ for $r = 0, 1, 2, \dots (k-1)$.
For $r = 0, 1, 2, \dots (k-1)$, $a_{k-r} = n-k-1$.
$\Rightarrow \dim(\Lambda \cap V_{n-k+k-r-(n-k-1)}) = \dim(\Lambda \cap V_{k-r+1})$
$= -\dim(\Lambda + V_{k-r+1}) + \dim(\Lambda) + \dim(V_{k-r+1})$
$\geq -(k+1) + k + (k-r+1) = k-r$. since $\Lambda + V_{k-r+1}$ is a vector space. $\Rightarrow \Lambda \in Z_k$. $\Rightarrow Z_k = \{\Lambda : \Lambda \subset V_{k+1}\}$.
Suppose $\Lambda \supset V_{k-r}$.
$\dim(\Lambda \cap V_{n-k+k-r-i-(n-k)}) = \dim(\Lambda \cap V_{k-r-i}) \geq \dim(V_{k-r-i})$