$\Rightarrow$ $\theta(z+\lambda) = e_\lambda(z)\cdot\theta(z)$ since $\varphi_{z+\lambda}\circ\varphi_z^{-1}(\theta(z))$
$= e_\lambda(z)\cdot\theta(z)$.
$\Rightarrow$ Since $e_\lambda(z) = 1$ for $\lambda = \lambda_\alpha$, and $e_\lambda(z) = e^{-2\pi i z_\alpha}$
for $\lambda = \lambda_{\alpha+n}$,

$$\theta(z+\lambda_\alpha) = \theta(z) \text{ and } \theta(z+\lambda_{n+\alpha}) = e^{-2\pi i z_\alpha}\theta(z).$$

Consider a section $\tilde{\theta}([z]) = [z, \theta(z)]$. $\Rightarrow$ By the relation $\theta(z+\lambda) = \theta(z)\cdot e_\lambda(z)$, $\tilde{\theta}$ is well-defined.

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Now if $\| \ \|$ is any metric on L, we can write

$$\|\tilde{\theta}(z)\|^2 = h(z)\cdot|\theta(z)|^2$$

for any section $\tilde{\theta}$ of L; evidently h will be a positive C^∞ function of z satisfying

$$h(z)\,|\theta(z)|^2 = \|\tilde{\theta}(z)\|^2 = h(z+\lambda)\,|\theta(z+\lambda)|^2$$

for any $\lambda\in\Lambda$: thus

$$h(z+\lambda_\alpha) = h(z),$$
$$h(z+\lambda_{n+\alpha}) = |e^{2\pi i z_\alpha}|^2\, h(z).$$

$\ulcorner$ $\|\tilde{\theta}(z)\|^2 = \|\tilde{\theta}([z])\|^2$ $\Rightarrow$ Not so precise.
$\|\pi^*\tilde{\theta}(z)\|^2 = h(z)\,|\theta(z)|^2 = \|\pi^*\tilde{\theta}(z+\lambda)\|^2 = h(z+\lambda)\,|\theta(z+\lambda)|^2$ $\Rightarrow$ $h(z)\,|\theta(z)|^2 = h(z+\lambda)\,|e_\lambda(z)\,\theta(z)|^2$
$\Rightarrow$ $h(z) = h(z+\lambda_\alpha)$, $h(z) = h(z+\lambda_{n+\alpha})\,|e^{-2\pi i z_\alpha}|^2$
Note: By the property of a metric, $h(z)$ is determined by the given metric, independent of the choice of a nonvanishing (locally) section θ.

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