Stop here. ⏎

Now return to our cubic surface $S \cong \widetilde{\mathbb{P}}^2_{p_1 \cdots p_6}$ in $\mathbb{P}^3$. Consider first the image of the exceptional divisors $E_1, E_2, \cdots E_6$ in S. Since $\tilde{C} \cdot E_i = 1$, we see that each of the curves E_i has degree 1 in $\mathbb{P}^3$, hence must be a line.

Γ $\tilde{C} = \pi^* 3H - E_1 - \cdots - E_6$. $\tilde{C} \cdot E_i = -E_i \cdot E_i = 1$.

$$\begin{tikzcd}
{[\tilde{C}]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & {[H]} \arrow[d] \\
\widetilde{\mathbb{P}}^2 \arrow[r, "\iota_{\tilde{C}}"] & \mathbb{P}^3
\end{tikzcd}$$

$$H \cdot \iota_{\tilde{C}}(E_i) = \tilde{C} \cdot E_i = 1$$
$$\parallel$$
$$(H \cap \iota_{\tilde{C}}(\widetilde{\mathbb{P}}^2)) \cap \iota_{\tilde{C}}(E_i) = \iota_{\tilde{C}}(\tilde{C}) \cdot \iota_{\tilde{C}}(E_i)$$

$\Rightarrow$ $\iota_{\tilde{C}}(E_i)$ has degree 1 in $\mathbb{P}^3$ $\Rightarrow$ By p174, it is a line. ⏎

Likewise, if F_{ij} $(j > i)$ is the proper transform in $\widetilde{\mathbb{P}}^2$ of the line $L_{ij} = \overline{p_i p_j}$ in $\mathbb{P}^2$, then

$$F_{ij} \cdot \tilde{C} = (\pi^* L_{ij} - E_i - E_j) \cdot (\pi^* 3H - \Sigma E_k)$$
$$= L_{ij} \cdot 3H - 2 = 3 - 2 = 1,$$

so that the image of F_{ij} in $S \subset \mathbb{P}^3$ is again a line; there are 15 such lines.