The metric connection D^* on E^* can be defined by the requirement

$$d(\tau(\sigma)) = \tau(D\sigma) + D^*\tau(\sigma). \quad \text{for } \sigma \in \mathcal{A}^0(E)(U)$$
$$\tau \in \mathcal{A}^0(E^*)(U).$$

Note:

$$E_U \xrightarrow{\varphi_U} U \times \mathbb{C}^n, \qquad e_j \longleftrightarrow \bar{e}_j \qquad e_j = \begin{array}{c} \\ j \\ \\ \end{array}\!\!\begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$E_U^* \xrightarrow{\varphi_U^{-1*}} U \times \mathbb{C}^{n*} \cong U \times \mathbb{C}^n.$$

$$e_i^* \longleftrightarrow \bar{e}_i^*$$

$$\Rightarrow \quad e_i^*(e_j) = \varphi_U^*\bar{e}_i^*(e_j) = \bar{e}_i^*(\varphi_U e_j) = \bar{e}_i^*(\bar{e}_j)$$

$$D^{*\prime\prime} = \bar{\partial} \;\; ?$$

$\tau = \sum f_i \tau_i \qquad \sigma = \sum g_j \sigma_j.$ where $\tau_i(\sigma_j) = \delta_{ij}$ and τ_i, σ_j's are holomorphic sections on E_U^*, E_U repectively.

$$\Rightarrow \quad \overset{(1,0)}{\partial(\tau(\sigma))} + \overset{(0,1)}{\bar{\partial}(\tau(\sigma))} = \overset{(1,0)}{\tau(D'\sigma)} + \overset{(0,1)}{\tau(D''\sigma)} + \underset{(1,0)}{D^{*\prime}\tau(\sigma)} + \underset{(0,1)}{D^{*\prime\prime}\tau(\sigma)}$$

$$\Rightarrow \quad \bar{\partial}(\tau(\sigma)) = \tau(D''\sigma) + D^{*\prime\prime}\tau(\sigma) = \tau(\bar{\partial}\sigma) + D^{*\prime\prime}\tau(\sigma)$$

$$\Rightarrow \quad (D^{*\prime\prime}\tau)(\sigma) = \bar{\partial}(\tau(\sigma)) - \tau(\bar{\partial}\sigma) \overset{?}{=} (\bar{\partial}\tau)(\sigma)$$

$$\Rightarrow \quad D^{*\prime\prime}\tau = \bar{\partial}\tau \qquad \Rightarrow \quad D^{*\prime\prime} = \bar{\partial}$$

We have to show $\bar{\partial}(\tau(\sigma)) - \tau(\bar{\partial}\sigma) = (\bar{\partial}\tau)(\sigma)$.

$$\tau(\sigma) = \sum f_i g_i \quad \Rightarrow \quad \bar{\partial}(\tau(\sigma)) = \sum(\bar{\partial} f_i\, g_i + f_i \bar{\partial} g_i)$$
$$= \sum \bar{\partial} f_i\, g_i + \sum f_i\, \bar{\partial} g_i = (\bar{\partial}\tau)(\sigma) + \tau(\bar{\partial}\sigma)$$