functions $f_\alpha \in \mathcal{O}(U_\alpha)$ such that the locus $(f_\alpha = 0) = V \cap U_\alpha$ and s.t for any α, β,

$$g_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta).$$

But since $H^1(\mathbb{C}^n, \mathcal{O}^*) = 0$, the cocycle ($- \equiv \div$, $+ \equiv \times$ since $\mathcal{O}^*$ is multiplicative)

$$\{g_{\alpha\beta}\} \in C^1(\underline{U}, \mathcal{O}^*)$$

is a coboundary.

$\Rightarrow$ After refinement of the open covering if necessary, $\exists$ a cochain $\{h_\alpha\} \in C^0(\underline{U}, \mathcal{O}^*)$ s.t

$$\frac{f_\alpha}{f_\beta} = g_{\alpha\beta} = \frac{h_\beta}{h_\alpha}.$$

The entire function $f = f_\alpha h_\alpha = f_\beta h_\beta$ has zero locus exactly V.

3. Application of the vanishing $H^q((\mathbb{C})^k \times (\mathbb{C}^*)^\ell, \mathcal{O}) = 0$.

To compute the cohomology groups $H^q(\mathbb{P}^1, \mathcal{O})$, take u and $v = 1/u$ Euclidean coordinates on $\mathbb{P}^1$,

$$U = \{u_0 \neq 0\} \qquad V = \{v_0 \neq 0\}, \qquad \mathbb{P}^1 = [(u_0, v_0)]$$

$$\phi \downarrow \cong \qquad \psi \downarrow \cong$$

$$\mathbb{C} \ni u = \frac{v_0}{u_0} \qquad \mathbb{C} \ni v = \frac{u_0}{v_0} \Rightarrow \psi \circ \phi^{-1}(u) = u^{-1} = v$$

$$\Rightarrow \quad U \cap V = \mathbb{C}^*.$$

Thus the cover $\{U, V\}$ of $\mathbb{P}^1$ is acyclic, i.e.

$$H^q(\mathbb{C}, \mathcal{O}) = H^q(U, \mathcal{O}) = H^q(V, \mathcal{O}) = H^q(U \cap V, \mathcal{O}) = H^q(\mathbb{C}^*, \mathcal{O}) = 0$$

for $q > 0$.