1. Using $T^* = T^{*\prime} \oplus T^{*\prime\prime}$, we can write $D = D' + D''$, with $D' : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{1,0}(E)$ & $D'' : \mathcal{A}^0(E) \longrightarrow \mathcal{A}^{0,1}(E)$.

$\Rightarrow$ We say that a connection D on E is compatible with complex structure if $D'' = \bar\partial$.

2. If E is hermitian, D is said to be compatible with the metric if $d_v\langle \xi, \eta\rangle = \langle D_v \xi, \eta\rangle + \langle \xi, D_{\bar v}\eta\rangle$. (refer to Poor) p268.

lemma: E hermitian. v.b. $\Rightarrow$ $\exists$ a unique connection D on E compatible with both the metric and the complex structure.

pf). $e = \{e_1, e_2 \cdots e_n\}$ holo. frame for E. $h_{ij} = \langle e_i, e_j\rangle$.
Suppose $\exists$ such D.

Its matrix θ w.r.t e, must have type $(1,0)$.

$\because$ $De_i = \theta_{ij} e_j = \theta_{ij}^{(1,0)} e_j + \theta_{ij}^{(0,1)} e_j \Rightarrow \bar\partial e_i = \theta_{ij}^{(0,1)} e_j = 0$

$\Rightarrow \theta_{ij}^{(0,1)} = 0 \Rightarrow De_i = \theta_{ij}^{(1,0)} e_j. \Rightarrow \theta_{ij} = \theta_{ij}^{(1,0)}$.

$\Rightarrow d h_{ij} = d(e_i, e_j) = \langle De_i, e_j\rangle + \langle e_i, De_j\rangle$ $\left(\langle e_i, De_j\rangle(v) = \langle e_i, D_{\bar v} e_j\rangle\right)$

$= \langle \sum_k \theta_{ik} e_k, e_j\rangle + \langle e_i, \sum \theta_{jk} e_k\rangle = \sum \theta_{ik} h_{kj} + \sum \overline{\theta_{jk}} h_{ik}$

$=$ type $(1,0)$ + type $(0,1)$. $\left(\overline{\theta}(v) = \overline{\theta(\bar v)} \right.$ see p263. Poor$\left.\right)$

Comparing types, we have

$$\partial h_{ij} = \sum \theta_{ik} h_{kj} \quad \text{i.e} \quad \partial h = \theta h$$
$$\bar\partial h_{ij} = \sum \overline{\theta_{jk}} h_{ik} \quad \text{i.e} \quad \bar\partial h = h\, {}^t\overline{\theta}.$$