$$\omega = \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} \qquad (g(z) \in \mathcal{O}(\overline{U}))$$

we let
$$\omega_t = \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_{t,1}(z) \cdots f_{t,n}(z)}.$$

If we assume that $f^{-1}(0)$ is a finite set of points interior to U, then $\|f(z)\| \geq \epsilon > 0$ on the boundary $\partial U = \overline{U} - U$, and so $\|f_t(z)\| \geq \epsilon/2 > 0$ for t sufficiently close to 0.

⊩ $t=0$ $f_0(z) = f(z)$. by tube lemma it is true since ∂U is compact. ⌟

Consequently $f_t^{-1}(0)$ will again be a finite set of points interior to U.

⊩ By the argument above, $\exists\, \delta > 0$ s.t $\|f_t(z)\| \geq \frac{\epsilon}{2}$ on ∂U for $0 \leq t \leq \delta$. Suppose $f_t^{-1}(0)$ is not a finite set of points interior to U for $0 \leq t \leq \delta$. $\Rightarrow$ $f_t^{-1}(0)$ is an algebraic variety of dim ≥ 1 in U. $\Rightarrow$ $f_1^{-1}(0)$ is compact $\Rightarrow$ By Liouvill theorem, contradiction as for a compact complex manifold. More clearly, a compact complex manifold can not be embedded into $\mathbb{C}^n$. ⌟

"For the reason of sufficiently small t, $f_t(z) = (1-t)z$".

On the other hand, by the explicit formula

$$\eta_{\omega_t} = C_n\, g(z)\, \frac{\sum (-1)^{i-1}\, \overline{f}_{i,t}\, d\overline{f}_{t,1} \wedge \cdots \widehat{d\overline{f}_{t,i}} \wedge \cdots \wedge d\overline{f}_{t,n} \wedge dz_1 \wedge \cdots \wedge dz_n}{\|f_t(z)\|^2}$$