Proof of the Garding Inequality.

We suppose that $\varphi_1, \cdots \varphi_n$ is a local unitary coframe for the hermitian metric, so that

$$ds^2 = \Sigma\, \varphi_i \otimes \overline{\varphi_i}\,.$$

A form of type $(p,q)$ is written locally as

$$\psi = \sum_{\substack{I,J \\ \text{"ordered"}}} \psi_{I,\bar J}\, \varphi_I \wedge \overline{\varphi_J} \qquad I = \{ i_1 < i_2 < \cdots i_p \}$$

There is a famous formular for the Laplacian, the Weitzenböck identity, which we shall use in the crude form

$$(\mathrm{W}). \qquad (\Delta \psi)_{I\bar J} = \Big( - \sum_{k=1}^{n} \nabla_k \nabla_{\bar k}\, \psi_{I\bar J} \Big) + A'(\psi)\,.$$

In other words, modulo lower-order terms the global Laplacian on forms looks like the Euclidean Laplacian $-\Sigma_k\, \partial^2/(\partial z_k\, \partial \bar z_k)$ on vector-valued functions.

The precise Weitzenböck formula identifies the lower-order terms. For a general hermitian metric, $A'(\psi)$ is a messy operator involving the torsion in its terms of first order. However, when the metric is Kähler, these drop out and $A'(\psi)$ is the algebraic operator.

$$A'(\psi)_{I\bar J} = \sum_{k,\, j_\alpha \in J} R_{\bar j_\alpha k}\, \psi_{I\, \bar j_1 \cdots \bar k \cdots \bar j_q}\,. \quad (k \text{ in } \alpha\text{th spot})$$

where $\quad R_{j\bar k} = \sum R^{\,i}_{ij\bar k} \quad$ is the Ricci curvature.