particular metric. By the commutativity of Δ_d and $\Pi^{p,q}$ and the fact that Δ_d is real, the harmonic forms satisfy

$$(*)\quad \left\{ \begin{aligned} \mathcal{H}^r(M) &= \bigoplus_{p+q=r} \mathcal{H}^{p,q}(M) \\ \mathcal{H}^{p,q}(M) &= \overline{\mathcal{H}^{q,p}(M)} . \end{aligned} \right.$$

⌈ $\eta \in \mathcal{H}^r(M) \Rightarrow \Delta_d \eta = 0$

$\Rightarrow \eta = \bigoplus_{p+q=r} \eta^{p,q} \qquad \Delta_d \eta = \oplus \Delta_d \eta^{p,q} = 0$

$\Leftrightarrow \Delta_d \eta^{p,q} = 0$. This proves the first.

$\eta \in \mathcal{H}^{p,q}(M)$, $\Delta_d \eta = 0$. $\overline{\Delta_d \eta} = \overline{\Delta_d}(\overline{\eta}) = \Delta_d(\overline{\eta}) = 0$

$\Rightarrow$ proves the second ⌋

On the other hand, for η, a closed form of pure degree (p,q),

$$\eta = \mathcal{H}(\eta) + dd^* G(\eta),$$

where the harmonic part $\mathcal{H}(\eta)$ also has pure type (p,q).

Thus $H^{p,q}(M) \cong \mathcal{H}^{p,q}(M)$.

⌈

$$\begin{array}{ccc} H^{p,q}(M) & \longrightarrow & \mathcal{H}^{p,q}(M) \\ \rotatebox{90}{$\in$} & & \\ \eta & \longmapsto & \mathcal{H}(\eta) \end{array}$$

If $\mathcal{H}(\eta) = 0$, $\eta = dd^*G(\eta) \in dA^*(M)$

$\Rightarrow \eta \in Z_d^{p,q}(M) \cap dA^*(M) \Rightarrow \eta = 0$

Given $\phi \in \mathcal{H}^{p,q}(M)$, s.t $\Delta_d \phi = 0$,

$\Delta_d \phi = 0 \Leftrightarrow d\phi = 0$ & $d^*\phi$. $\Rightarrow \phi \in H^{p,q}(M)$.

Combining this with $(*)$ and the Hodge theorem (See P89)

$H^*_{DR}(M) \cong \mathcal{H}^*(M)$ for the Laplacian Δ_d,

we obtain the famous