Since '~' operation is some sort of lifting, the map is holomorphic.
$\Rightarrow \varphi : \tilde{\Delta} \longrightarrow \pi^{-1}\Delta$ is holomorphic, and one to one generically. $\Rightarrow$ Since $\varphi(\tilde{\Delta})$ is an analytic subvariety of $\pi^{-1}\Delta$, by the identity theorem, $\varphi(\tilde{\Delta}) = \pi^{-1}\Delta$, and φ is isomorphic.
More precisely, $\varphi : \mathbb{P}^{n-1} \longrightarrow \mathbb{P}^{n-1}$ is onto & holomorphic.

$\Rightarrow$ Let $d = \deg \varphi$, $0 \le d < \infty$.
Choose a point $p \in \mathbb{P}^{n-1} \subset \tilde{\Delta}$. $\varphi(p) \in \mathbb{P}^{n-1} \subset \pi^{-1}\Delta$.
$\Rightarrow$ We can choose open sets U, V s.t
U & V are biholomorphic to open sets in $\mathbb{C}^n$.
$p \in U \subset \tilde{\Delta}$, $V \subset \pi^{-1}\Delta$.
for $\varphi : U \longrightarrow V$, $\varphi^{-1}(\varphi(p)) = \{p\}$.
$\Rightarrow$ By the results on p667, since φ is one to one generically on U, φ is isomorphic into V.
Suppose $d \ge 2$.
$\Rightarrow \exists\ U' \ni p'$ s.t $\varphi(p') = \varphi(p)$ $U' \cap U = \phi$
$\varphi : U' \longrightarrow V$, $\varphi^{-1}(\varphi(p)) = \{p'\}$.
(If necessary, choose V smaller.)
$\Rightarrow$ For $q \in V - \mathbb{P}^{n-1}$, $\exists$ distinct p_0, p_0' s.t