$g \in \mathcal{O}^*(U)$, clearly $g + I^3$ is invertible in $\mathcal{O}(U)/I^3(U)$.

$\Rightarrow \phi$ is onto. Suppose $\phi(g+I^3) = \cdot\,(1+I^3) = g\hat{\ }(1+I^3)$
$\Rightarrow g - 1 \in I^3 \Rightarrow g + I^3 = 1 + I^3. \Rightarrow \phi$ is one to one.
$\phi((g+I^3)(h+I^3)) = \phi(gh + I^3) = gh\hat{\ }(1+I^3) = g(1+I^3)\cdot h(1+I^3) \Rightarrow \phi$ is homomorphism $\Rightarrow$ Since ϕ is one to one & onto, ϕ is isomorphism.

$$\Rightarrow H^0(\mathbb{P}^2, \mathcal{O}^*_{(2)}) \longrightarrow \underset{\substack{\| \\ H^1(\mathbb{P}^2, \mathcal{O}(-3H)) \\ \| \\ 0}}{H^1(\mathbb{P}^2, 1+I^3)} \longrightarrow \underset{\substack{\| \\ H^1(\mathcal{O}^*_{\mathbb{P}^2})}}{H^1(\mathbb{P}^2, \mathcal{O}^*)} \longrightarrow \underset{\substack{\| \\ H^1(\mathcal{O}^*_{(2)})}}{H^1(\mathbb{P}^2, \mathcal{O}^*_{(2)})}$$

$$\longrightarrow \underset{\substack{\| \\ H^2(\mathbb{P}^2, \mathcal{O}(-3H)) \\ \| \\ \mathbb{C}}}{H^2(\mathbb{P}^2, 1+I^3)} \longrightarrow \underset{\substack{\| \\ 0 \\ \| \\ H^2(\mathcal{O}^*_{\mathbb{P}^2})}}{H^2(\mathbb{P}^2, \mathcal{O}^*)}.$$

$$\Rightarrow \quad 0 \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}^*) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}^*_{(2)}) \xrightarrow{\psi} \mathbb{C} \longrightarrow 0.$$
$$\mathcal{L}_{(2)} \longmapsto \alpha$$

$\Rightarrow$ For $\mathcal{L}_{(2)}$ to be the restriction of some $\mathcal{L} \in H^1(\mathbb{P}^2, \mathcal{O}^*)$, $\psi(\mathcal{L}_{(2)}) = 0$ must be valid. $\lrcorner$

Assuming now this to be the case, we have $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(n)$ $(n > 0)$, and the exact sheaf sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}((n-3)H) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(nH) \longrightarrow \mathcal{O}(\mathcal{L}_{(2)}) \longrightarrow 0$$

together with $H^1(\mathcal{O}_{\mathbb{P}^2}((n-3)H)) \cong H^1(\mathcal{O}_{\mathbb{P}^2}(-nH)) = 0$ gives in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}((n-3)H)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(nH)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(\mathcal{L}_{(2)})) \to 0.$$