is commutative.

Proof. For an index set $J \subset (1, \cdots r)$, we let $J^\circ = (1, \cdots r) - J$ be the complementary index set and define

$$e_{J^*} = \pm e_{J^\circ} \in E_{r-k}.$$

The sign is chosen to make $e_J \wedge e_{J^*} = e_1 \wedge \cdots \wedge e_r$. Then we define $\hat{e}_J \in \mathrm{Hom}_{\mathcal{O}}(E_k, \mathcal{O})$ by

$$\hat{e}_J(e_{J'}) = \begin{cases} 0, & J \neq J', \\ 1, & J = J'. \end{cases}$$

The isomorphism in the lemma is given by

$$\hat{e}_J \longrightarrow e_{J^*}.$$

It is a direct computation that the diagram is commutative. Q.E.D.

$\ulcorner$ $e_{J^*} = \epsilon \, e_{J^\circ}$

$\Rightarrow$ $e_J \wedge \epsilon e_{J^\circ} = e_1 \wedge \cdots \wedge e_r = \epsilon \, e_J \wedge e_{J^\circ}$.

$\Rightarrow$ $\epsilon = \mathrm{sign}\,(J, J^\circ)$. $J = \{j_1 < j_2 < \cdots < j_k\}$, $J^\circ = \{j^\circ_1 < \cdots < j^\circ_{r-k}\}$

$$\partial^* \hat{e}_J = \sum_{\#I = k+1} a_I \, \hat{e}_I.$$

$$\begin{aligned}(\partial^* \hat{e}_J)(e_I) = a_I &= \hat{e}_J(\partial e_I) \\ &= \hat{e}_J\left(\sum (-1)^{\nu-1} f_{i_\nu} \, e_{i_1} \wedge \cdots \wedge \hat{e}_{i_\nu} \wedge \cdots \wedge e_{i_{k+1}}\right) \\ &= \sum (-1)^{\nu-1} f_{i_\nu} \, \hat{e}_J(e_{i_1} \wedge \cdots \wedge \hat{e}_{i_\nu} \wedge \cdots \wedge e_{i_{k+1}})\end{aligned}$$