Finally, the $\bar\partial$- Laplacian on E is defined by

$$\Delta = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : \quad A^{p,q}(E) \longrightarrow A^{p,q}(E).$$

An E-valued form φ is called harmonic if $\Delta\varphi = 0$. (Again, harmonic forms φ are exactly the forms of smallest norm in their Dolbeault cohomology class $\varphi + \bar\partial A^{p,q-1}(E)$.) We let

$\mathcal{H}^{p,q}(E) = \operatorname{Ker}\Delta$ be the harmonic space.

Now, the analytic part of the proof of the Hodge theorem for the $\bar\partial$ - Laplacian on ordinary differential forms on M is essentially local: we can always find appropriate solutions of $\Delta\varphi = 0$ in the completion of $A^{p,q}(M)$ in the L_2-norm; the problem is to show that these solutions are in fact C^∞. Writing out E-valued forms in terms of a frame for E, all the local estimates used in the proof of the Hodge theorem for $A^*(M)$ go over to $A^{p,q}(E)$ – the only difference is that in each estimate we will get lower-order terms involving the coefficient functions for the metric on E as well as the metric on T^*M, and these can be estimated out as before. Thus the Hodge theorem holds for the $\bar\partial$- Laplacian on E, that is:

1. $\mathcal{H}^{p,q}(E)$ is finite dimensional, and
2. If $\mathcal{H}$ denotes the orthogonal projection $A^{p,q}(E) \to \mathcal{H}^{p,q}(E)$, there exists an operator

$$G : A^{p,q}(E) \longrightarrow A^{p,q}(E)$$

s.t $\quad G(\mathcal{H}^{p,q}(E)) = 0,$