$$\sigma_2 = a_{12} e_1 + \cdots + a_{k2} e_k$$
$$\vdots \qquad \vdots$$
$$\sigma_n = a_{1n} e_1 + \cdots + a_{kn} e_k.$$

For simplicity, let $k=2$, $n=3$.

$$\sigma_1 = a_{11} e_1 + a_{21} e_2 \quad \cdots \text{①}$$
$$\sigma_2 = a_{12} e_1 + a_{22} e_2 \quad \cdots \text{②}.$$
$$\sigma_3 = a_{13} e_1 + a_{23} e_2 \quad \cdots \text{③}.$$

$$\Rightarrow \quad \text{①} \times a_{12} = a_{12}\sigma_1 = a_{11}a_{12} e_1 + a_{21}a_{12} e_2$$
$$- \text{②} \times a_{11} = a_{11}\sigma_2 = a_{11}a_{12} e_1 + a_{11}a_{22} e_2$$
$$a_{12}\sigma_1 - a_{11}\sigma_2 = (a_{21}a_{12} - a_{11}a_{22}) e_2.$$

Let's look at $a_{21}a_{12} - a_{11}a_{22} = f$ closely.
$\Rightarrow$ f is holomorphic over U.
If f is zero over U, $\sigma_1 = l\sigma_2$ for some constant l. $\Rightarrow$ Since σ_1, & σ_2 are holomorphic, $\sigma_1 = l\sigma_2$ over M $\Rightarrow$ Contradiction to the fact that σ_1 & σ_2 are linearly independent each other.
Thus f can not be 0 over U. $\Rightarrow$ If $x_0 \in U$ s.t $f(x_0) \neq 0$,

$$\Rightarrow \quad e_2 = \frac{a_{12}}{f}\sigma_1 - \frac{a_{11}}{f}\sigma_2$$

over $V \ni x_0$, $f \neq 0$.

$\Rightarrow$ e_1, e_2 can be expressed as a linear combination of σ_1, σ_2 over some open set $V \subset U$ where $f \neq 0$ over V.
$\Rightarrow$ Given a section σ over M, consider the restriction $\sigma|V$. $\sigma|V$ is expressed as a linear combination of e_1, e_2, $\Rightarrow$ $\sigma|V$ can be