$G(z_1, 0) = (z_1, z_2) \Rightarrow (z_1, z_2) = (g_1(z_1), g_2(z_1))$,,

$\Rightarrow z_2 = g_2(z_1) \Rightarrow z_2 = z_2(z_1)$

$\Rightarrow$ Locally, $z_1 \longmapsto (z_1, g_2(z_1)) = (z_1, z_2(z_1))$ is biholomorphic. $\Rightarrow$ z_1 will serve as local coordinate on S. $\Rightarrow$

$$\begin{tikzcd}
\pi_p : S \subset \mathbb{P}^2 \arrow[r] & \mathbb{P}^1 \quad \text{near } q \\
{[Z_0, Z_1, Z_2]} \arrow[r, mapsto] \arrow[d, "Z_1/Z_0"'] & {[Z_0, Z_1]} \arrow[d] \\
z_1 \in \mathbb{C}^1 \subset \mathbb{C}^2 \arrow[r] & z_1 \in \mathbb{C}^1
\end{tikzcd}$$

$\Rightarrow$ $\pi_p(z_1) = z_1$ locally. $\Rightarrow$ π_p is not ramified. $\lrcorner$

Consequently the order of vanishing of $\partial z_1/\partial z_2$ at q — that is, the ramification index $v(q)$ of the map π_p at q minus one — is equal to the order of zero of $\partial f/\partial z_2$ at $q \in S$ — that is, the multiplicity of intersection of S with the curve $(\partial f/\partial z_2 = 0)$ at q.

$\ulcorner$ If $(\partial f/\partial z_2)(q) = 0$, $(\partial f/\partial z_1)(q) = 0$ since $\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right) \neq 0$ at q.

$$\Rightarrow \frac{\partial f}{\partial z_2} + \frac{\partial f}{\partial z_1}\frac{\partial z_1}{\partial z_2} = \left(\frac{\partial f}{\partial z_1} \quad \frac{\partial f}{\partial z_2}\right) \cdot \left(\frac{\partial z_1}{\partial z_2}, \quad 1\right) \equiv 0 \text{ on } S.$$

In this case, $\pi_p : S \longrightarrow \mathbb{P}^1$

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathbb{P}^1 \arrow[d] \\
\mathbb{C} \arrow[r] & \mathbb{C} \\
z_2 \arrow[r, mapsto] & z_1(z_2)
\end{tikzcd}$$