$$\begin{tikzcd} \mathcal{O}^{(k)} \arrow[r, "\alpha"] \arrow[d] & M \arrow[d] \arrow[l, dashed, bend left, "\gamma"] \\ \mathcal{O}^{(k)}/m\mathcal{O}^{(k)} \arrow[r, "\alpha_0"] & M/mM \arrow[l, dashed, bend left, "\gamma_0"] \end{tikzcd}$$

Given $y \in \mathcal{O}^{(k)}/m\mathcal{O}^{(k)}$, $\alpha_0(y) = x \in M/mM$. Since $\alpha \circ \gamma =$ identity, $\alpha_0 \circ \gamma_0 =$ identity. $\Rightarrow$ $\alpha_0 \circ \gamma_0(x) = x$. $\Rightarrow$ Since $\alpha_0(\gamma_0(x)) = x$ & $\alpha_0(y) = x$, by the injectiveness of α_0, $y = \gamma_0(x)$. $\Rightarrow$ γ_0 is surjective. $\Rightarrow$ By Nakayama lemma, γ is surjective. From $\alpha \circ \gamma = id$, γ is injective, and so γ is isomorphic. $\Rightarrow$ α is isomorphic.

Clearly, k is not minimal $\Rightarrow$ $\alpha_0 : \underset{\cong\ \mathbb{C}^k}{\dfrac{\mathcal{O}^{(k)}}{m\mathcal{O}^{(k)}}} \longrightarrow M/mM$ is not isomorphic. ∎

By Nakayama's lemma (third form) γ is surjective and α is an isomorphism.

Note that the definition of projective may be reprased as follows:

$$K \xrightarrow{f} L \longrightarrow 0$$
$$\Downarrow$$
$$\mathrm{Hom}_{\mathcal{O}}(M, K) \xrightarrow[f_*]{} \mathrm{Hom}_{\mathcal{O}}(M, L) \longrightarrow 0.$$

Given $\alpha \in \mathrm{Hom}_{\mathcal{O}}(M, L)$, $\exists$ $\beta \in \mathrm{Hom}_{\mathcal{O}}(M, K)$ s.t $f \circ \beta = \alpha$. $\Rightarrow$ $f_* \beta = \alpha$ $\Rightarrow$ f_* is onto. Conversely, easy to prove. ∎

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