$\Rightarrow$ Since $f^*(\tilde{\sigma}_a) = \widetilde{f_*(\sigma_a)}$,

$$\tilde{\sigma}_{a^*} \wedge \tilde{\sigma}_{b^*} = \sum \delta(a,b,c)\, \tilde{\sigma}_{c^*}.$$

$$\Rightarrow \quad \sigma_{a^*} \cdot \sigma_{b^*} = \sum \delta(a,b;c)\, \sigma_{c^*}$$

$$\Rightarrow \quad \delta(a,b;c) = \delta(a^*, b^*; c^*).$$

Note that

$$\ell(a^*) = a_1 \quad \text{and} \quad a_1^* = \ell(a)$$

so that the formulas 1 and 2 above are, as expected, equivalent under the $*$ map.

Γ Suppose $\ell(a) = \ell$.

$\Rightarrow \quad a_1 \geq a_2 \geq \cdots \geq a_\ell \neq 0 > 0 = \cdots$

$\Rightarrow \quad a^*_{a_\ell} \geq \ell \quad \Rightarrow \quad a^*_{a_\ell} = \ell.$

$$a_1^* \geq a_2^* \geq \cdots \geq a^*_{a_\ell} = \ell.$$

Since a^* is the smallest sequence, $a_1^* = \cdots = a^*_{a_\ell} = \ell$

$\Rightarrow \ \ell = a_1^* \quad \Rightarrow \quad \ell(a) = a_1^*$.

Since a is dual to a^*, a is the smallest sequence s.t. $a_{a_i^*} \geq i$.

$\Rightarrow$ By the same reasoning above, $\ell(a^*) = a_1$.

In a different way. if $a_k \neq 0$, & $a_{k+1} = 0$.

$$a_1^* \geq \ \geq a^*_{a_k} = k \geq a^*_{a_{k-1}} = k-1 > \cdots \geq a^*_{a_1} = 1 > \underset{\overset{\shortparallel}{0}}{a^*_{a_1-1}}$$

since a^* is the smallest.

$\Rightarrow \ \ell(a^*) = a_1$.