∂ - Poincare lemma. $H_{\partial}^{p,q}(\Delta) = 0$, $p \geq 1$.

$\partial \eta' = 0$. $\eta' \in A^{1,0}(loc) \Rightarrow \exists\ \gamma$ s.t $\partial \gamma = \eta'$, $\gamma \in C^{\infty}$.

$\Rightarrow \varphi = \frac{1}{2}(\gamma + \bar{\gamma})$ is real. since $\overline{\varphi(\phi)} = \frac{1}{2}\overline{(\gamma + \bar{\gamma})(\phi)}$

$$= \frac{1}{2}\overline{\gamma(\phi) + \bar{\gamma}(\phi)} = \frac{1}{2}\left(\overline{\gamma(\phi)} + \gamma(\bar{\phi})\right) = \frac{1}{2}\left(\bar{\gamma}(\bar{\phi}) + \gamma(\bar{\phi})\right)$$

$$= \frac{1}{2}(\gamma + \bar{\gamma})(\bar{\phi}). \Rightarrow \frac{\sqrt{-1}\,\partial\bar{\partial}}{2}(\gamma + \bar{\gamma})$$

$$= \frac{\sqrt{-1}}{2}\left(-\bar{\partial}\eta' + \partial\overline{\eta'}\right) = \frac{1}{2}T + \frac{\sqrt{-1}}{2}\partial\overline{\eta'} = \frac{1}{2}T - \frac{1}{2}\overline{\sqrt{-1}\,\bar{\partial}\eta'}$$

$$= \frac{1}{2}T + \frac{1}{2}\overline{-\sqrt{-1}\,\bar{\partial}\eta'} = \frac{1}{2}T + \frac{1}{2}\overline{T} = \frac{1}{2}T + \frac{1}{2}T$$

$$= T$$

Using the fact that $\sqrt{-1}\,\partial\bar{\partial}\varphi$ is a distribution of order zero, it may be proved that φ is a locally L^1 function, but we will not completely prove this, since we do not need it. Intuitively, the argument is that

$$\Delta\varphi = \sqrt{-1}\,\Lambda\,\partial\bar{\partial}\varphi$$

is a distribution of order zero, hence it is (more or less) in the Sobolev space H_0. By the regularity theorem, then, φ is (more or less) in the Sobolev space H_2.

From $\Lambda\partial - \partial\Lambda = \sqrt{-1}\,\bar{\partial}^*$ key steps

$$\Lambda\partial\bar{\partial} - \partial\Lambda\bar{\partial} = \sqrt{-1}\,\bar{\partial}^*\bar{\partial} = \sqrt{-1}\,\Delta$$

$\Rightarrow \sqrt{-1}\Lambda\partial\bar{\partial}\varphi - \sqrt{-1}\partial\Lambda\bar{\partial}\varphi = -\Delta\varphi$ is zero order $\Rightarrow \Delta\varphi \in H_0$

$\Rightarrow$ By the regularity theorem, $\varphi \in H_2$. $\Rightarrow \varphi \in \underset{H_0}{\underset{\parallel}{L^2}} \Rightarrow \varphi \in L^1$ locally

see P85 ~ P86