To evaluate this expression, consider the meromorphic functions f, g on the Riemann sphere given in terms of a Euclidean coordinate by

$$f(z) = \prod_{k=0}^{n} (\alpha_k - z), \qquad g(z) = z^n;$$

then $(g(z)/f(z))\, dz = \varphi$ is a meromorphic differential with simple poles at $z = \alpha_k$ and $z = \infty$, and

$$Res_{\alpha_i}(\varphi) = \frac{-\alpha_i^n}{\prod_{j \neq i} (\alpha_j - \alpha_i)},$$

$$Res_{\infty}(\varphi) = (-1)^n.$$

$\ulcorner$ Clearly at $z = \alpha_k$, $\frac{g(z)}{f(z)} dz$ has a simple pole.

$$Res_{\alpha_i}(\varphi) \cdot 2\pi\sqrt{-1} = \int_{B_\epsilon(\alpha_i)} \frac{g(z)}{f(z)} dz$$

$$= -\int_{B_\epsilon(\alpha_i)} \frac{1}{z - \alpha_i} \frac{z^n}{\prod_{j \neq i} (\alpha_j - z)} dz$$

$$= -2\pi\sqrt{-1} \frac{\alpha_i^n}{\prod_{j \neq i} (\alpha_j - \alpha_i)} \quad \text{by Cauchy formula}$$

$$\Rightarrow \quad Res_{\alpha_i}(\varphi) = -\frac{\alpha_i^n}{\prod_{j \neq i} (\alpha_j - \alpha_i)}$$