Proof. Let $\pi : \mathcal{O}/I(f') \longrightarrow \mathcal{O}/I(f)$ be the natural projection, and write

$$f_1' = \sum_i b_i f_i,$$

so that $f' = Af$, where

$$A = \begin{bmatrix} b_1, & b_2, & \cdots & b_n \\ 0, & 1 & \cdots & 0 \\ & & & \vdots \\ 0, & 0, & \cdots & 1 \end{bmatrix}.$$

If $g = \sum_i c_i f_i$ is in the ideal $I(f)$, then

$$b_1 g = c_1 \Big(\sum_i b_i f_i \Big) + \sum_{i \geq 2} (b_1 c_i - c_1 b_i) f_i$$

is in the ideal $I(f')$.

$$\begin{aligned} b_1 g &= b_1 c_1 f_1 + b_1 c_2 f_2 + \cdots + b_1 c_n f_n \\ &= c_1 (b_1 f_1 + b_2 f_2 + \cdots + b_n f_n) + b_1 c_2 f_2 - c_1 b_2 f_2 + b_1 c_3 f_3 \\ &\quad - c_1 b_3 f_3 + \cdots \qquad + b_1 c_n f_n - c_1 b_n f_n \\ &= c_1 \sum_i b_i f_i + \sum_{i \geq 2} (b_1 c_i - c_1 b_i) f_i \\ &= c_1 f_1' + \sum_{i \geq 2} (b_1 c_i - c_1 b_i) f_i \in I(f'). \end{aligned}$$

Thus multiplication by b_1 gives a map

$$\alpha : \mathcal{O}/I(f) \longrightarrow \mathcal{O}/I(f')$$

going in the opposite direction to π.

$$\alpha : \mathcal{O}/I(f) \longrightarrow \mathcal{O}/I(f')$$

$$g + I(f) \longmapsto b_1 g + I(f').$$ well-defined since if $g \in I(f)$, then $b_1 g \in I(f')$.