4. The log complex. We now come to an interesting situation. Suppose M is a complex manifold and D a divisor on M. We say that D has normal crossings in case $D = \sum_\nu D_\nu$, where the irreducible components D_ν of D are smooth and meet transversely. At a point p through which k of the D_ν pass, we may choose local holomorphic coordinates $(z_1, \cdots z_n)$ in a neighborhood $U = \{ |z_i| < 1 \}$ of $p = (0, 0, \cdots 0)$ such that

$$D \cap U = \{ z_1 \cdots z_k = 0 \}$$

is the union of coordinate hyperplanes.

⌈ Meet transversely means that

if $D_{\nu_1} = (f_1 = 0)$ $D_{\nu_2} = (f_2 = 0)$, $\cdots$ $D_{\nu_k} = (f_k = 0)$,

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f_1}{\partial z_1}, & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \\ \frac{\partial f_k}{\partial z_1}, & \cdots & \frac{\partial f_k}{\partial z_n} \end{pmatrix} = k .$$

$\Rightarrow$ Suppose
$$\det \begin{pmatrix} \frac{\partial f_1}{\partial z_1}, & \cdots & \frac{\partial f_1}{\partial z_k} \\ & & \\ \frac{\partial f_k}{\partial z_1}, & \cdots & \frac{\partial f_k}{\partial z_k} \end{pmatrix} \neq 0 .$$

Consider the following map, locally

$$\mathbb{C}^n \xrightarrow{\phi} \mathbb{C}^n$$
$$(z_1, \cdots z_n) \longmapsto (f_1, \cdots f_k, z_{k+1}, \cdots z_n)$$

$$\Rightarrow J(\phi) = \left(\begin{array}{c|c} \left(\frac{\partial f_i}{\partial z_j}\right) & 0 \\ \hline 0 & I \end{array} \right) \quad 1 \le i, j \le k . \Rightarrow \det(J(\phi)) \neq 0 \text{ at } p.$$