check that $\eta = d^c \log \|z\|^2$ restricts to the generator of $H^1(S^1)$ for each fiber.

Pf $\eta = d^c \log \|z\|^2$.

Let $n = 3$. $S^5 \subset \mathbb{C}^3$
$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$$

Fix $(z_1^\circ, z_2^\circ, z_3^\circ) \in S^5$, & consider $\{(\alpha z_1^\circ, \alpha z_2^\circ, \alpha z_3^\circ) \mid |\alpha| = 1\}$.

$$\Rightarrow \quad S^1 \xrightarrow{\phi} S^5$$
$$\alpha \longmapsto \alpha(z_1^\circ, z_2^\circ, z_3^\circ).$$

$$\phi^*\eta = \phi^* \frac{\sqrt{-1}}{4\pi}(\bar\partial - \partial)\log\|z\|^2.$$

$$= \phi^* \frac{\sqrt{-1}}{4\pi}\bar\partial \log\|z\|^2 - \phi^*\frac{\sqrt{-1}}{4\pi}\partial\log\|z\|^2$$

Since $\bar\partial \log\|z\|^2 = z_1 d\bar z_1 + z_2 d\bar z_2 + z_3 d\bar z_3$ and $\partial\log\|z\|^2 = \bar z_1 dz_1 + \bar z_2 dz_2 + \bar z_3 dz_3$,

$$\frac{\sqrt{-1}}{4\pi}\phi^* d^c \log\|z\|^2 = \frac{\sqrt{-1}}{4\pi}(\alpha \bar z_1^\circ z_1^\circ d\bar\alpha + \alpha \bar z_2^\circ z_2^\circ d\bar\alpha + \alpha \bar z_3^\circ z_3 d\bar\alpha - \bar\alpha \bar z_1^\circ z_1^\circ d\alpha - \bar\alpha \bar z_2^\circ z_2 d\alpha - \bar\alpha \bar z_3^\circ z_3 d\alpha)$$

$$= \frac{\sqrt{-1}}{4\pi}(\alpha d\bar\alpha - \bar\alpha d\alpha)$$

$$= \frac{\sqrt{-1}}{4\pi}\{(x + \sqrt{-1}y)(dx - \sqrt{-1}dy) - (x - \sqrt{-1}y)(dx + \sqrt{-1}dy)\}$$

$$= \frac{1}{2\pi}(x\,dy - y\,dx) = \frac{d\theta}{2\pi}, \quad \theta = \tan\frac{y}{x}.$$

This shows that $\eta|_{\text{each fiber}}$ is the generator.

$\Rightarrow d_2\eta = d\eta = dd^c\log\|z\|^2 = \omega$ is the standard Kähler form on $\mathbb{P}^n$. //

So, in this case the relation $d_2\eta = \omega$