for all $\eta \in C^\infty(T)$. In case the weak solution φ is also a C^∞ function, the Laplacian is self-adjoint, meaning that $\langle \eta, \Delta_d \varphi \rangle = \langle \eta, \psi \rangle$ for all $\eta \in C^\infty(T)$, and so $\Delta_d \varphi = \psi$ in the usual ~~co~~ sense. Weak solutions are easy to find by Hilbert-space methods, and the point is to prove regularity.

First, note that the weak solutions of the homogeneous equation $\Delta_d \varphi = 0$ satisfy

$$\langle \|\xi\|^2 e^{i\langle \xi, x\rangle}, \varphi \rangle = 0 \quad \text{for all } \xi.$$

$$\langle \underset{\|}{\Delta_d} e^{i\langle \xi, x\rangle}, \varphi \rangle = \langle e^{i\langle \xi, x\rangle}, \Delta_d \varphi \rangle$$

Thus the weak ~~solution~~ harmonic space consists of the constant functions, defined by $\varphi_\xi = 0$ for $\xi \neq 0$.

Suppose φ is a weak solution of $\Delta_d \varphi = 0$.

$$\langle 0, \psi \rangle = \langle \varphi, \Delta_d \psi \rangle \quad \text{for all } \psi \in C^\infty(T).$$

For $\psi = e^{i\langle \xi, x\rangle}$, $\langle \varphi, \Delta_d e^{i\langle \xi, x\rangle} \rangle = 0$. for $\boxed{\begin{matrix}\xi = 0\\ \xi \neq 0\end{matrix}}$

$\Rightarrow \langle \varphi, e^{i\langle \xi, x\rangle} \cdot \|\xi\|^2 \rangle = 0 \Rightarrow \langle \varphi, e^{i\langle \xi, x\rangle} \rangle = 0$. for $\xi \neq 0$

$\Rightarrow \varphi_\xi = \int_T \varphi(x) e^{-i\langle \xi, x\rangle} dx \underset{\text{for } \xi \neq 0}{= 0}. \Rightarrow \varphi_\xi = 0. \Rightarrow \varphi_0 = ?$

Next, we observe that (*) make sense when $\psi \in L^2(T) = H_0$. A necessary condition for it to have a weak solution is that $\psi_0 = 0$, i.e ψ should be orthogonal to the harmonic space.

Let $\varphi = \Sigma \varphi_\xi e^{i\langle \xi, x\rangle}$, $\eta = \Sigma \eta_\xi e^{i\langle \xi, x\rangle}$, $\psi = \Sigma \psi_\xi e^{i\langle \xi, x\rangle}$.

$\Rightarrow \langle \Delta_d \eta, \varphi \rangle = \langle \eta, \psi \rangle \Rightarrow \langle \Sigma \varphi_\xi e^{i\langle \xi, x\rangle}, \Delta(\Sigma \eta_\xi e^{i\langle \xi, x\rangle}) \rangle = \langle \Sigma \psi_\xi e^{i\langle \xi, x\rangle}, \Sigma \eta_\xi e^{i\langle \xi, x\rangle} \rangle$