Kodaira - Nakano Vanishing Theorem

If $L \longrightarrow M$ is a positive line bundle, then

$$H^q(M, \Omega^p(L)) = 0 \quad \text{for } p+q > n.$$

pf) By hypothesis we can find a metric in L whose curvature form Θ is $2\pi/i$ times the associated $(1,1)$-form of a Kähler metric; let the metric on M be the one given by $\omega = \frac{i}{2\pi}\Theta$.

∵ By the def. of a positive line bundle, $\exists$ a metric in L s.t $\frac{i}{2\pi}$ times the curvature form is a positive

$(1,1)$-form. $\Rightarrow$ By p29, this positive $(1,1)$-form defines a hermitian metric, which is Kähler. //

Now by harmonic theory

$$H^q(M, \Omega^p(L)) \cong H^{p,q}_{\bar\partial}(M, L) \cong \mathcal{H}^{p,q}(L)$$

$$H^{p,q}_{\bar\partial}(M,L) = H^{p,q}_{\bar\partial}(L)$$

To prove the result, we will show that there are no nonzero harmonic L-valued forms of degree larger than n. We do this by interpreting the curvature operator $\Theta\eta = \Theta \wedge \eta$ alternately as $\left(\frac{i}{2\pi}\right)^{-1} L(\eta)$ and as $D^2\eta$, where D is the metric connection on L, and using the basic identity above.

∵ $\Theta\eta = \Theta\wedge\eta \qquad \frac{2\pi}{i}L(\eta) = \frac{2\pi}{i}\omega\wedge\eta = \Theta\wedge\eta$

see also p 401

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