$p \in L \Rightarrow \exists\ U$ open in M s.t

$$\begin{tikzcd} U \arrow[r, "\varphi"] & \mathbb{C}^n \\ U \cap L \arrow[u] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{C}^k \arrow[u, "\text{inclusion}"'] \end{tikzcd}$$

Since N is a smooth hypersurface of M, $\exists\ g$ holomorphic function on U (maybe we take smaller nbd of p) s.t

$$\{ q \in U \mid g(q) = 0 \} = U \cap N.$$

Suppose $\dfrac{\partial g \circ \varphi^{-1}}{\partial z_1} \neq 0$.

$\Rightarrow$ $\begin{array}{ccc} \mathbb{C}^n & \longrightarrow & \mathbb{C}^n \\ (z_1 \cdots z_n) & \overset{f}{\longmapsto} & (g\circ\varphi^{-1}(z), z_2 \cdots z_n) \end{array}$ is a biholomorphic map.

$$\Rightarrow \begin{tikzcd} & U \arrow[r, "\varphi"] & \mathbb{C}^n \arrow[r, "f"] & \mathbb{C}^n \\ U \cap N \arrow[ur] & & (z_1, z_2 \cdots z_n) \arrow[r, mapsto] & (g\circ\varphi^{-1}(z), z_2, \cdots z_n) \\ & U \cap L \arrow[ul] \arrow[uu] \arrow[r, "\varphi|"] & \mathbb{C}^k \ni (z_1, z_2, \cdots z_k) \arrow[uu] & \end{tikzcd}$$

$\Rightarrow$ Since $\varphi|^{-1}(z_1, z_2, \cdots z_k) \in U \cap L \subset U \cap N$,

$$g\circ\varphi^{-1}(z_1, z_2 \cdots z_k, 0 \cdots 0) = 0.$$

$\Rightarrow$ $f\circ\varphi|_{U\cap L} = \{(0, z_2, \cdots z_k, 0 \cdots 0)\} \subset \mathbb{C}^{k-1}$

$\Rightarrow$ Contradiction, since $f\circ\varphi|_{U \cap L} \cong \mathbb{C}^k$.

$\Rightarrow$ $\dfrac{\partial g\circ\varphi^{-1}}{\partial z_i} = 0$ for $1 \le i \le k$.

Without loss of generality, $\dfrac{\partial g\circ\varphi^{-1}}{\partial z_{k+1}} \neq 0$