It follows from this calculation that:

1. If $\ell(X)$ is any linear functional on $\mathbb{C}^{n+1}$, the vector field

$$v(X) = \ell(X)\frac{\partial}{\partial X_i}$$

descends to $\mathbb{P}^n$ — that is, $\pi_* v(X) = \pi_* v(\lambda X)$ for any $X \in \mathbb{C}^{n+1}$, $\lambda \in \mathbb{C}$. We will denote by $\pi_* v$ the induced vector field on $\mathbb{P}^n$.

$$\pi_* v(X) = \ell(X)\,\pi_*\left(\frac{\partial}{\partial X_i}\right)$$

$$\pi_* v(\lambda X) = \lambda \ell(X)\,\pi_*\left(\frac{\partial}{\partial X_i}\Big|_{(X_0, \lambda X_1, \dots)}\right)$$

$$\Rightarrow \pi_*\left(\frac{\partial}{\partial X_1}\right) = \frac{1}{X_0}\frac{\partial}{\partial x_i} \quad \text{at } (X_0 \dots$$

$$\pi_*\left(\frac{\partial}{\partial X_1}\Big|_{(\lambda X_0, \dots)}\right) = \frac{1}{\lambda X_0}\frac{\partial}{\partial x_i} = \frac{1}{\lambda}\pi_*\left(\frac{\partial}{\partial X_1}\Big|_{(X_0 \dots)}\right)$$

$$\pi_*\left(\frac{\partial}{\partial X_0}\Big|_{(X_0, \dots)}\right) = -\sum \frac{X_i}{X_0^2}\frac{\partial}{\partial x_i}$$

$$\pi_*\left(\frac{\partial}{\partial X_0}\Big|_{(\lambda X_0, \dots \lambda X_n)}\right) = -\sum \frac{\lambda X_i}{\lambda^2 X_0^2}\frac{\partial}{\partial x_i} = -\frac{1}{\lambda}\sum \frac{X_i}{X_0^2}\frac{\partial}{\partial x_i}$$

$$= +\frac{1}{\lambda}\pi_*\left(\frac{\partial}{\partial X_0}\Big|_{(X_0 \dots)}\right)$$

$$\Rightarrow \pi_* v(X) = \ell(X)\,\pi_*\left(\frac{\partial}{\partial X_i}\right)$$

$$\pi_* v(\lambda X) = \lambda \ell(X)\,\frac{1}{\lambda}\pi_*\left(\frac{\partial}{\partial X_i}\Big|_{(X_0 \dots)}\right) = \ell(X)\,\pi_*\left(\frac{\partial}{\partial X_i}\Big|_{(X_0 \dots)}\right)$$

Here $\ell : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ linear

2. The tangent space $T_x'(\mathbb{P}^n)$ to $\mathbb{P}^n$ at a point $x = \pi(X)$

MOOKEUK CO.