On the other hand, we may choose our open set $\mathcal{U}$ a ball around the point $p$; and the standard contraction $z \longmapsto tz$ of $\mathcal{U}$ onto $p$ induces, via $\pi$, a contraction of $\tilde{\mathcal{U}}$ onto $E$. Thus we have

$$H_i(\tilde{M}) = H_i(M) \oplus H_i(E), \quad i > 0.$$

Suppose $i \neq 0,\ 2n-1,\ 2n-2$.

Since $\tilde{\mathcal{U}} \simeq E \cong \mathbb{P}^{n-1}$, $\mathcal{U} \simeq *$, and $\mathcal{U}^* \simeq S^{2n-1}$,

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"] & H_i(\mathbb{P}^{n-1}) \oplus H_i(\tilde{M}^*) \arrow[r] \arrow[d, "\pi_* \cong"] & H_i(\tilde{M}) \arrow[r] \arrow[d, "\pi_*"] & 0 \arrow[d, "\cong"] \\
0 \arrow[r] & H_i(M^*) \arrow[r] & H_i(M) \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad H_i(\tilde{M}) = H_i(\tilde{M}^*) \oplus H_i(E) = H_i(M^*) \oplus H_i(E) = H_i(M) \oplus H_i(E)$$

Second $i = 2n-1 > 0$.

$$\begin{tikzcd}
H_i(\tilde{\mathcal{U}}^*) = \mathbb{Z} \arrow[r, "f"] \arrow[d, "\cong"] & \underset{\substack{\shortparallel \\ 0}}{H_i(\mathbb{P}^{n-1})} \oplus H_i(\tilde{M}^*) \arrow[r] \arrow[d, "\pi_* \cong"] & H_i(\tilde{M}) \arrow[r] \arrow[d, "\pi_*"] & 0 \arrow[d, "\cong"] \\
H_i(\mathcal{U}^*) \cong \mathbb{Z} \arrow[r, "g"] & \underset{\substack{\shortparallel \\ 0}}{H_i(\mathcal{U})} \oplus H_i(M^*) \arrow[r] & H_i(M) \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad H_i(\tilde{M}) \cong H_i(\tilde{M}^*)/\operatorname{im} f \cong H_i(M^*)/\operatorname{Im} g \cong H_i(M) \oplus \underset{\substack{\shortparallel \\ 0.}}{H_i(E)}$$

Third $i = 2n-2 > 0$