Since moreover nH is the unique line bundle of degree n on $\mathbb{P}^1$, it follows that every irreducible nondegenerate curve of degree n in $\mathbb{P}^n$ is projectively isomorphic to the rational normal curve.

$$\begin{tikzcd}
H \arrow[d] & H|_{\tilde{C}} \arrow[l, "\Gamma"'] \arrow[d] & \bar{i}_{nH}^{*}(H|_{\tilde{C}}) \overset{?}{=} \arrow[l] \arrow[d] & i^{*}H = H|_{C} \arrow[r] \arrow[d] & H \arrow[d] \\
\mathbb{P}^{n} & \bar{i}_{nH}(\mathbb{P}^1) \arrow[l, hook'] & \mathbb{P}^1 \arrow[l, "i_{nH}"'] \arrow[r, "q"] & C \arrow[r, hook, "i"] & \mathbb{P}^{n}
\end{tikzcd}$$

$\bar{i}_{nH}(\mathbb{P}^1) = \tilde{C}$

Since the degree of C is n, $c_1(H|_C) = c_1(i^*H) = n$.
This implies $q^*(H)$ is a bundle of degree n,

$$\mathbb{P}^1 \xrightarrow{q} C \Rightarrow q^*: H^2(C, \mathbb{R}) \xrightarrow{\cong} H^2(\mathbb{P}^1, \mathbb{R})$$
$$c_1(H|_C) \longmapsto q^* c_1(H|_C)$$

In the similar way, we get a line bundle $\bar{i}_{nH}^{*}(H|_{\tilde{C}})$ of degree n, since $\tilde{C}$ is a curve of degree n.

$\Rightarrow$ By P145, since nH is the only line bundle of degree n, $\exists$ an isomorphism $\bar{i}_{nH} \circ q^{-1} : C \to \tilde{C}$ s.t

$$\begin{tikzcd}
H|_C \arrow[r, "(\bar{i}_{nH}\circ q^{-1})^{*}"] \arrow[d] & H|_{\tilde{C}} \arrow[d] \\
C \arrow[r, "\bar{i}_{nH}\circ q^{-1}"'] & \tilde{C}
\end{tikzcd}$$
commutative

Thus again by P177, C is projectively isomorphic to $\tilde{C}$.