($\Leftarrow$) Clear. $\lrcorner$

The argument also makes it pretty clear that residues will be complicated when $p \geq 3$.
We shall now show that

For a p-form φ of the second kind, given any point $x_0 \in M$ there is a meromorphic $(p-1)$-form ψ such that $\varphi - d\psi = \eta$ is holomorphic near x_0. The converse is true when $p=1$.

Proof. Given $x_0 \in M$, we may find an ample divisor D not passing through x_0, and then for $U = M - D$ by the algebraic de Rham theorem,

$$H^*_{DR}(U) \cong H^*_{DR}(U, alg).$$

Definition: A divisor D in M is said to be ample if there is a positive integer m such that $[mD] \longrightarrow M$ is very ample.
By Bertini's theorem, we may choose $\mathbb{P}^{N-1}$ s.t $\mathbb{P}^{N-1} \cap M = D$ is smooth and transversely and $D \not\ni x_0$, where $M \subset \mathbb{P}^N$. (See P162).
$\Rightarrow$ $[D] \longrightarrow M$ is positive line bundle $\Rightarrow$ By the Kodaira embedding theorem, D is ample.
Suppose D is a ample divisor in M.
$\Rightarrow$ $[kD]$ is positive. $\Rightarrow$ $k\eta_D$ is positive $(1,1)$-form $\Rightarrow$ η_D is positive $(1,1)$-form $\Rightarrow$ $[D] \longrightarrow M$ is positive line bundle. $\Rightarrow$ We can apply the algebraic deRham