$\circledast$ holds since measure $(\mathbb{P}^{n-2})$ is zero in $\mathbb{P}^{n-1}$.

Wrong here !

$$= - C_n \int_{|z_1| \le r} \frac{dz_1}{z_1} \wedge \bar\partial_{z_1} \int_{U_0} \varphi \cdot \left(\frac{\sum d\omega_i \wedge d\bar\omega_i}{1+|\omega|^2} - \frac{(\sum \bar\omega_i d\omega_i) \wedge (\sum \omega_i d\bar\omega_i)}{(1+|\omega|^2)^2} \right.$$

Cauchy integral formula

$$= C_n \int_{|z_1| = r} \frac{dz_1}{z_1} \int_{U_0} \varphi \, (\omega_i, d\bar\omega_i, d\omega_i) - 2\pi\sqrt{-1}\, C_n \, \varphi(0)$$

Again, here we have encountered the technigical difficulty. I think that the argument on P62 is the best. So let's leave this way, sometime later we can see this problem again. ⌋

Smoothing and Regularity

A distribution $T \in \mathcal{D}(\mathbb{R}^n)$ is said to be smooth in case $T = T_\psi$ for a C^∞ function $\psi(x)$ on $\mathbb{R}^n$. ⌈ See P368 ⌋
We shall now make precise the sense in which the smooth distributions are dense among all distributions.

Let $\chi(x) \in C_c^\infty(\mathbb{R}^n)$ be a nonnegative function supported in a neighborhood of the origin, with

$$\int_{\mathbb{R}^n} \chi(x)\, dx = 1 .$$

In a little while we shall assume that χ is radially symmetric, i.e., in polar coordinates $x = r\omega$

$$\chi(x) = \chi(r) .$$

⌈ See Hirsch, P for the radially symmetric χ. ⌋