$$\mathbb{P}(N_{L/X}) = \mathbb{P}^1 \times \mathbb{P}^1$$

and it follows that $n = 0$, i.e., the normal bundle of L in X is trivial.

┌ By P799, $\tilde{f}_L(F) = Q$ and $\tilde{f}$ is one-to-one $\Rightarrow$ $F = \pi^{-1}(L) \cong Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ by P479 since Q is smooth.

$c_2(\mathbb{P}^1 \times \mathbb{C}^2) = 0$, but $c_2(H \oplus H^{-1}) = c_1(H) \cdot c_1(H^{-1}) + c_1(H) \cdot c_1(H^{-1}) = -2$ $\Rightarrow$ $H \oplus H^{-1}$ has no nonvanishing section while $\mathbb{P}^1 \times \mathbb{C}^2$ has one. In other words, the intersection pairing of those two spaces are different. $\Rightarrow$ $\mathbb{P}^1 \times \mathbb{C}^2 \not\cong H \oplus H^{-1}$. Here we used the fact that the top Chern class (= Euler class) is the obstruction of existence of a nonvanishing section. $\Rightarrow$ $n = 0$. Refer to P411 ~ P 413.

┘

On the other hand, if L is special, then $\mathbb{P}(N_{L/X})$ is the ruled surface $S_2 = \mathbb{P}(H^m \oplus H^{m+2})$, and so

$$N_{L/X} = H \oplus H^{-1}.$$

┌ 'Make-up'

$\mathbb{P}(N_{L/X}) = \mathbb{P}^1 \times \mathbb{P}^1$. $\quad N_{L/X} = H^n \oplus H^{-n}$

$\Rightarrow$ $\mathbb{P}(N_{L/X}) = \mathbb{P}((H^n \otimes (H^{-n})^*) \oplus \mathbb{C}_{\mathbb{P}^1}) = \mathbb{P}((H^n \otimes H^n) \oplus \mathbb{C}_{\mathbb{P}^1}) =$

by P517