By the definition of δ, if we set

$$z_{\alpha\beta\gamma} = h_{\alpha\beta} + h_{\beta\gamma} - h_{\alpha\gamma} = (\delta h)_{\alpha\beta\gamma}$$
$$= \frac{1}{2\pi i}(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma}),$$

then $\{z_{\alpha\beta\gamma}\} \in Z^2(\mathcal{U}, \mathbb{Z})$ is a cocycle representing $c_1(L)$.

Now choose any connection D on L. In terms of the frame $e_\alpha(z) = \varphi_\alpha^{-1}(z,1)$ on U_α, D is given by its connection matrix, which in this case is a 1-form θ_α. As was worked out in Section 5 of Chapter 0, in $U_\alpha \cap U_\beta$

$$\theta_\alpha = g_{\alpha\beta}\,\theta_\beta\, g_{\alpha\beta}^{-1} + d g_{\alpha\beta} \cdot g_{\alpha\beta}^{-1},$$

i.e

$$\theta_\beta - \theta_\alpha = -g_{\alpha\beta}^{-1}\, d g_{\alpha\beta} = -d(\log g_{\alpha\beta})$$

and the curvature matrix is the global 2-form

$$\Theta = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha = d\theta_\beta .$$

$\ulcorner$ since $\Theta_\alpha = g_{\alpha\beta}\,\Theta_\beta\, g_{\alpha\beta}^{-1} = \Theta_\beta$ $\lrcorner$

Since Θ is given as a closed 2-form and $c_1(L)$ is given as a Čech cocycle, we must look at the explicit form of the de Rham isomorphism. From the proof of de Rham's theorem, we have exact sequences of sheaves

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{Z}_d^1 \to 0, \quad 0 \to \mathcal{Z}_d^1 \to \mathcal{A}^1 \to \mathcal{Z}_d^2 \to 0$$

, giving us boundary isomorphisms

$$\frac{H^0(\mathcal{Z}_d^2)}{d H^0(\mathcal{A}^1)} \xrightarrow{\delta_1} H^1(\mathcal{Z}_d^1), \quad H^1(\mathcal{Z}_d^1) \xrightarrow{\delta_2} H^2(\mathbb{R}).$$

To calculate $\delta_1(\Theta)$, we write Θ locally as $d\theta_\alpha$: