This implies that $P_1, P_2, \cdots P_7$ impose 7-linearly independent conditions on conics. Contradiction to the assumption. ⌟

Let L_{ij} denote the line $\overline{P_i P_j}$. The cubic curve

$$L_{23} + L_{45} + L_{67}$$

contains $P_2, \cdots P_7$ and hence P_1 as well; thus P_1 is collinear with two other points P_i, which we may take to be P_2 and P_3.

Suppose now that the line $L = \overline{P_2 P_3 P_1}$ also contains one of the points $P_4, \cdots P_7$, say P_4. Then since the cubic

$$L_{25} + L_{36} + L_{47}$$

contains P_1, we must have either P_5, P_6 or P_7 lying on L as well; thus we have five collinear points.

⌜ If $P_1 \in L_{25}$, then $\overline{P_1 . P_2 P_5} = L_{25} = L$. ⇒

We can see that either P_5, P_6 or P_7 must lie on L.

⇒ Thus we have five collinear points. ⌟

If, on the other hand, none of the points $P_4, P_5, \cdots P_7$ lies on the line L, then since the cubics

$$L_{24} + L_{35} + L_{67}, \quad L_{24} + L_{36} + L_{57}, \quad \text{and} \quad L_{25} + L_{36} + L_{47}$$

all contain P_1 but the lines L_{24}, L_{25}, L_{35} and L_{36}