As we can see there P377. there is no restriction to δ except small enough. If $V \subset W \subset U$, $C_c^\infty(V) \subset C_c^\infty(W)$. $\Rightarrow$ $\psi_W|_V = \psi_V$.

Consequently there is a harmonic function ψ in U such that $T = T_\psi$.

"Comment"

$\Delta T = 0 \Rightarrow T_\delta = T_{\delta'}$, even if $\delta \neq \delta'$.

For, $(T_{\epsilon_1})_{\epsilon_2}(y) = \int_{\mathbb{R}^n} T_{\epsilon_1}(x)\, \chi_{\epsilon_2}(y-x)\, dx = T_{\epsilon_1}(y)$

since T_{ϵ_1} is harmonic, by P377.

$\Rightarrow$ ψ_V is the same for all ϵ, because $\psi_V = T_\epsilon$ for some ϵ. "

As an application we have the Regularity for the $\bar{\partial}$-operator. If $U \subset \mathbb{C}^n$ is an open set and $T \in \mathcal{D}(U)$ satisfies $\bar{\partial} T = 0$, then $T = T_f$ for some $f \in \mathcal{O}(U)$.

Proof. By one of our Hodge identies from Section ? of Chapter 0,

$$\Delta = \sqrt{-1}\ \Lambda \partial \bar{\partial}$$

on $\mathbb{C}^n$.

By P111, we have $\Lambda \partial - \partial \Lambda = \sqrt{-1}\, \bar{\partial}^*$.

$\Rightarrow$ $\sqrt{-1}\ \Lambda \partial \bar{\partial} = \sqrt{-1}\ (\partial \Lambda + \sqrt{-1}\ \bar{\partial}^*)\, \bar{\partial} = \sqrt{-1}\ \partial \Lambda \bar{\partial} - \bar{\partial}^* \bar{\partial}$.