for $g=0$ above, f/g must be constant again.

③ order $= -1$ $(a_{-3} = a_{-2} = 0,\ a_{-1} \neq 0)$

$\frac{\sigma}{s_0}$ is holomorphic away from p by the argument above.

$\Rightarrow \frac{\sigma}{s_0}$ is constant $\Rightarrow \sigma = c\, s_0$. But we cannot have s_0 on S with $g(S)=1$ by the argument above.

Thus a meromorphic function on S holomorphic on $S-\{p\}$ and of order ≥ -3 at p is determined by its principal part, up to constant. In other words, it depends on the values of a_{-3}, a_{-2} and a_0.

$\Rightarrow \dim H^0(S, \mathcal{O}(L)) \leq 3 \Rightarrow$ Since $h^0(S, \mathcal{O}(L)) \geq 3$,
$h^0(S, \mathcal{O}(L)) = 3. \Rightarrow \iota_L : S \longrightarrow \mathbb{P}^2$.

It is worthwhile, however, to go through the process explicitly in this case. First we shall establish a basic general fact:

Lemma (Residue Theorem). For φ a meromorphic one-form on a compact Riemann surface S with polar divisor $a_1 + \cdots + a_d$,

$$\sum_i \operatorname{Res}_{a_i}(\varphi) = 0.$$

proof.) Letting $B_\epsilon(a_i)$ be an ϵ-disc around a_i, we have by Stokes' theorem

$$0 = -\int_{S - \bigcup_i B_\epsilon(a_i)} d\varphi \overset{\text{Stokes' theorem.}}{=} \int_{\partial(\bigcup_i B_\epsilon(a_i))} \varphi = \sum_i \operatorname{Res}_{a_i}(\varphi).$$

Q.E.D

$\int_{B_\epsilon(a_i)} \varphi = 2\pi i \operatorname{Res}_{a_i}(\varphi)$. $\Rightarrow$ It follows.