

Let $\eta \in H^{p,q}(L)$ be a harmonic form. Then

$$\mathbb{H} = D^2 = \bar{\partial} D' + D' \bar{\partial}, \quad \text{so from } \bar{\partial} \eta = 0$$

$$\mathbb{H} \eta = \bar{\partial} D' \eta,$$

and

$$\begin{aligned} 2i \langle \mathbb{H} \eta, \eta \rangle &= 2i \langle \bar{\partial} D' \eta, \eta \rangle = 2i \langle (\bar{\partial} \Lambda - \frac{i}{2} D'^*) D' \eta, \eta \rangle \\ &= \langle D'^* D' \eta, \eta \rangle = \langle D' \eta, D' \eta \rangle \geq 0, \end{aligned}$$

$$\text{Since } \langle \bar{\partial} \Lambda D' \eta, \eta \rangle = \langle \Lambda D' \eta, \bar{\partial}^* \eta \rangle = 0.$$

$$\begin{aligned} \text{If } \mathbb{H} &= D^2 = (D' + \bar{\partial})(D' + \bar{\partial}) = D' D' + \bar{\partial} D' + D' \bar{\partial} + \bar{\partial}^2 \\ &= D'^2 + \bar{\partial} D' + D' \bar{\partial} \end{aligned}$$

But since \mathbb{H} is (1,1)-form, $D'^2 = 0$.

$$\Rightarrow \mathbb{H} = \bar{\partial} D' + D' \bar{\partial}.$$

$$\Delta \eta = 0 \Leftrightarrow \bar{\partial} \eta = 0 \text{ \& \> } \bar{\partial}^* \eta = 0.$$

$$\Rightarrow \mathbb{H} \eta = \bar{\partial} D' \eta.$$

$$\begin{aligned} i \langle \mathbb{H} \eta, \eta \rangle &= i \langle \bar{\partial} D' \eta, \eta \rangle = i \langle (\bar{\partial} \Lambda - \frac{i}{2} D'^*) D' \eta, \eta \rangle \\ &= i \langle \bar{\partial} \Lambda D' \eta, \eta \rangle + \langle D'^* D' \eta, \eta \rangle = i \langle \Lambda D' \eta, \bar{\partial}^* \eta \rangle \\ &+ \langle D' \eta, D' \eta \rangle = 0 + \langle D' \eta, D' \eta \rangle \geq 0. \quad \text{—} \end{aligned}$$

$$\begin{aligned} \text{Similarly, } 2i \langle \mathbb{H} \eta, \eta \rangle &= 2i \langle D' \bar{\partial} \eta, \eta \rangle = \\ 2i \langle D' (\bar{\partial} \Lambda + \frac{i}{2} D'^*) \eta, \eta \rangle &= - \langle D' D'^* \eta, \eta \rangle = - \langle D'^* \eta, D'^* \eta \rangle \\ &\leq 0. \end{aligned}$$

$$\begin{aligned} \text{If } i \langle \mathbb{H} \eta, \eta \rangle &= i \langle (\bar{\partial} D' + D' \bar{\partial}) (\Lambda \eta), \eta \rangle \\ &= i \langle \bar{\partial} D' \Lambda \eta, \eta \rangle + i \langle D' \bar{\partial} \Lambda \eta, \eta \rangle = i \langle D' \Lambda \eta, \bar{\partial}^* \eta \rangle \\ &+ i \langle D' \bar{\partial} \Lambda \eta, \eta \rangle = 0 + i \langle D' \bar{\partial} \Lambda \eta, \eta \rangle \\ &= i \langle D' (\bar{\partial} \Lambda + \frac{i}{2} D'^*) \eta, \eta \rangle = i \langle D' \bar{\partial} \Lambda \eta, \eta \rangle - \langle D' D'^* \eta, \eta \rangle \\ &= - \langle D' D'^* \eta, \eta \rangle = - \langle D'^* \eta, D'^* \eta \rangle \leq 0. \quad \text{—} \end{aligned}$$

$$\text{Combining, } 2i \langle [\Lambda, \mathbb{H}] \eta, \eta \rangle \geq 0.$$