⊢ $\overline{H,p} \cap (S - \{p\}) \xrightarrow{\pi_p} H \cap \pi_p(S)$ one to one onto.

Choose $H \subset \mathbb{P}^{N-1}$ so that H does not contain the singular points of $\pi_p(S)$ and H intersects $\pi_p(S)$ transversally. (maybe redundant)

$q \in \overline{H,p} \cap (S - \{p\})$.

$\Rightarrow q = [q_0, q_1, \cdots q_{N-1}, a]$, not all q_i's zero. $[q_0, \cdots q_{N-1}] \in H$

$\Rightarrow \pi_p(q) = [q_0, \cdots q_{N-1}] \in H \quad \Rightarrow \pi_p(q) \in H \cap \pi_p(S)$

Conversely $[q_0, \cdots q_{N-1}] \in H \cap \pi_p(S)$

$\Rightarrow [q_0, \cdots q_{N-1}, b] \in S$, not all q_i's zero.

$\Rightarrow$ If $b \neq 0$, $[q_0, \cdots q_{N-1}, b] \in (S - \{p\}) \cap \overline{H,p}$.

$\Rightarrow$ If $b = 0$, $[q_0, \cdots q_{N-1}, 0] \in \overline{H,p} \cap (S - \{p\})$.

since, obviously, $[q_0, \cdots q_{N-1}, 0] = p$.

π_p is one to one since we choose H generically and π_p is one to one except for a finite number of points. $\Rightarrow \deg \pi_p(S) = \#(\pi_p(S) \cap H)$ $= \#(\overline{H,p} \cap (S - \{p\})) = \#(\overline{H,p} \cap S) - 1 = \deg S - 1$. Here we can choose H generically so that $\overline{H,p} \cap S$ is smooth, see p208 }. ⊣

The Riemann - Hurwitz Formula

We know from elementary topology that a compact Riemann surface S has only one topological invariant, which we may take to be the genus

$$g(S) = \frac{b_1(S)}{2} = \frac{-\chi(S) + 2}{2},$$