$[-d-1,\ a_d-d-1]$, and so exactly one of these intervals will fail to contain an integer c_i-i.

Pf

$$\sigma_{a_1,\cdots a_{j-1},\, a_{j+1}-1,\cdots a_d-1}\cdot\sigma_{a_j+d-j}=\sigma_{a_j+d-j}\cdot\sigma_{a_1,\cdots a_{j-1},\, a_{j+1}-1,\cdots a_d-1}.$$

$$=\sum_{\substack{b_i\leq c_i\leq b_{i-1}\\ \sum c_i=a+\sum b_i,}}\sigma_c \qquad \begin{array}{ll} b_i=a_i \cdots & \text{for } i\leq j-1 \\ b_i=a_{i+1}-1 & \text{for } i\geq j. \\ a=a_j+d-j. & \end{array}$$

$$a_1\leq c_1\leq a_0=n-k \Rightarrow c_1\in[a_1,\ n-k]$$

$$\Rightarrow c_1-1\in[a_1-1,\ n-k]$$

$$a_2\leq c_2\leq a_1 \Rightarrow c_2-2\in[a_2-2,\ a_1-2]$$

$$a_3\leq c_3\leq a_2 \Rightarrow c_3-3\in[a_3-3,\ a_2-3]$$

$$\vdots \qquad\qquad \vdots$$

$$a_{j-1}\leq c_{j-1}\leq a_{j-2} \Rightarrow c_{j-1}-(j-1)\in[a_{j-1}-(j-1),\ a_{j-2}-(j-1)]$$

$$a_{j+1}-1\leq c_j\leq a_{j-1} \Rightarrow c_j-j\in[a_{j+1}-(j+1),\ a_{j-1}-j]$$

$$a_{j+2}-1\leq c_{j+1}\leq a_{j+1}-1 \Rightarrow c_{j+1}-(j+1)\in[a_{j+2}-(j+2),\ a_{j+1}-(j+2)]$$

$$\vdots \qquad\qquad \vdots$$

$$a_d-1\leq c_{d-1}\leq a_{d-1}-1 \Rightarrow c_{d-1}-(d-1)\in[a_d-d,\ a_{d-1}-d]$$

$$0\leq c_d\leq a_d-1 \Rightarrow c_d-d\in[-d,\ a_d-(d+1)]$$

$$\Rightarrow -d+c_d\in[-(d+1),\ a_d-d-1]$$

$$0\leq c_d\leq a_{d-1} \Rightarrow c_d-d\in[-d,\ a_{d-1}-d]\cap[-(d+1),\ a_d-(d+1)]$$

$\Rightarrow$ We have $[a_1-1,\ n-k]$,
$[a_2-2,\ a_1-2]$