Let $U' \subset U$ be the locus of $\left|\left(\frac{\partial f_i}{\partial z_j}\right)_{1\le i,j\le k}\right| \neq 0$,
($\Rightarrow$ U' is an open subset of U.)
and V' the locus $f_1 = \cdots = f_k = 0$. Then $V'' = V' \cap U'$
is a complex submanifold of U', and so of $\mathbb{C}^n$.
For any f_i ($f_i = 0$ on V), $i \geq k+1$, the differential $df_i \equiv 0$ on
$V' \cap U'$, i.e., f_i is constant on $V' \cap U' = V''$.
"For, let $(z_1^\circ, \cdots z_n^\circ) \in V'' = V' \cap U'$. Then we have
only to show that $df = 0$ at $(z_1^\circ, \cdots z_n^\circ) = q$ near p
Consider the following map ($\because$ $\{\left|\left(\frac{\partial f_i}{\partial z_j}\right)\right|_{1\le i,j\le k}\}$ is of measure zero).

$$(z_1, z_2, \cdots z_n) \xrightarrow{F} (f_1, f_2, \cdots f_k, z_{k+1}, \cdots z_n).$$

$\Rightarrow$ $|J(F)| \neq 0$ at $(z_1^\circ, \cdots z_n^\circ)$.
$\Rightarrow$ By the inverse function theorem, $\exists$ G locally s.t

$$G(f_1, f_2, \cdots f_k, z_{k+1}, \cdots z_n) = (z_1, z_2, \cdots z_n)$$

$\Rightarrow$ On V'' locally $G(0, \cdots 0, z_{k+1}, \cdots z_n) = (z_1, \cdots z_2, \cdots z_n)$.

$\Rightarrow$
$$z_1 = g_1(z_{k+1}, \cdots z_n)$$
$$z_2 = g_2(z_{k+1}, \cdots z_n)$$
$$\vdots$$
$$z_k = g_k(z_{k+1}, \cdots z_n)$$

$$\frac{\partial f_i(g_1, \cdots g_k, z_{k+1}, \cdots z_n)}{\partial z_{k+1}}$$

$$= \frac{\partial f_i}{\partial z_1} \cdot \frac{\partial g_1}{\partial z_{k+1}} + \frac{\partial f_i}{\partial z_2} \cdot \frac{\partial g_2}{\partial z_{k+1}} + \cdots + \frac{\partial f_i}{\partial z_k} \frac{\partial g_k}{\partial z_{k+1}} + \frac{\partial f_i}{\partial z_{k+1}} = ?$$

From $f_1(g_1, \cdots g_k, z_{k+1}, \cdots z_n) = f_2(g_1, \cdots g_k, z_{k+1}, \cdots z_n) = \cdots$