Comment : $[(p_0, p_1, p_2) + t(v_0, v_1, v_2)]$ is a line, $t \in \mathbb{C} \cup \{\infty\}$.

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^2$$

$$[X_0, X_1] \longmapsto [X_0(p_0, p_1, p_2) + X_1(v_0, v_1, v_2)] \Rightarrow \text{clearly onto.}$$

$$[X_0(p_0, p_1, p_2) + X_1(v_0, v_1, v_2)] = [Y_0(p_0, p_1, p_2) + Y_1(v_0, v_1, v_2)]$$

$$\Rightarrow \quad \begin{aligned} p_0 X_0 + v_0 X_1 &= p_0 Y_0 + v_0 Y_1 \\ p_1 X_0 + v_1 X_1 &= p_1 Y_0 + v_1 Y_1 \\ p_2 X_0 + v_2 X_1 &= p_2 Y_0 + v_2 Y_1 \end{aligned}$$

$\Rightarrow$ We get
$$\begin{aligned} X_1(p_1 v_0 - v_1 p_0) &= tY_1(v_0 p_1 - p_0 v_1) \\ X_0(p_0 v_1 - p_1 v_0) &= tY_0(p_0 v_1 - p_1 v_0) \\ X_1(v_1 p_2 - v_2 p_1) &= tY_1(v_1 p_2 - v_2 p_1) \\ X_0(p_1 v_2 - v_1 p_2) &= tY_0(p_1 v_2 - v_1 p_2). \end{aligned}$$

Unless $v_0 p_1 = v_1 p_0$, $X_1 = tY_1$, $tY_0 = X_0 \Rightarrow [X_0, X_1] = [Y_0, Y_1]$.

4. Kodaira Embedding Theorem

Line bundles and Maps to Projective Space.

We will be concerned in this section with determining exactly when a compact complex manifold is an algebraic variety, i.e., when it can be embedded in projective space. We first establish a basic formalism for maps to $\mathbb{P}^N$.

Let M be a compact complex manifold, $L \to M$ a holomorphic line bundle. Recall that to any subspace E of the vector space $H^0(M, \mathcal{O}(L))$ is associated the linear system

$$|E| = \{(s)\}_{s \in E} \subset \mathrm{Div}(M).$$