sequence of complexes.

Proofs. Since we have proved that the kernel of any surjective map $\mathcal{O}^{(k)} \longrightarrow M \longrightarrow 0$ is a finitely generated $\mathcal{O}$-module, assertion 1 follows.

$\ulcorner$ By P.179. $\pi_1 : \mathcal{O}^{(k)} \longrightarrow M$

$(g_1 \cdots g_k) \longmapsto g_1 f_1 + \cdots + g_k f_k$. $M = \{f_1, \cdots f_k\}$.

$\Rightarrow$ $\ker \pi_1$ is finitely generated.

$$0 \longrightarrow \ker \pi_1 \longrightarrow \mathcal{O}^{(k_1)} \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow \ker \pi_2 \longrightarrow \mathcal{O}^{(k_2)} \longrightarrow \ker \pi_1 \longrightarrow 0$$

$\Rightarrow$ Combining these two exact sequences, we have the following exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker \pi_2 \arrow[r] & \mathcal{O}^{(k_2)} \arrow[rr] \arrow[dr] & & \mathcal{O}^{(k_1)} \arrow[r] & M \arrow[r] & 0 \\ & & & \ker \pi_1 \arrow[ur] & & & \end{tikzcd}$$

Again since $\ker \pi_2$ is finitely generated, $\exists$ an exact sequence

$$0 \longrightarrow \ker \pi_3 \longrightarrow \mathcal{O}^{(k_3)} \longrightarrow \ker \pi_2 \longrightarrow 0.$$

Continue to combine the exact sequences, and we get an exact sequence which is projective.

$$\underset{E_n}{\overset{\shortparallel}{\mathcal{O}^{(k_n)}}} \longrightarrow \underset{E_{n-1}}{\overset{\shortparallel}{\mathcal{O}^{(k_{n-1})}}} \longrightarrow \cdots \longrightarrow \underset{E_0}{\overset{\shortparallel}{\mathcal{O}^{(k_1)}}} \longrightarrow M \longrightarrow 0.$$

Point: Don't be confused or misunderstood. $\lrcorner$

The proofs of assertions 2-4 are all similar, but with increasing complexity of notation. We shall therefore only