Now to the proof of local duality. The idea is to directly verify the statement for ideals $\{z_1^{k_1}, \cdots z_n^{k_n}\}$, and then to use the transformation law to deduce the result for the ideal $\{f_1, \cdots f_n\} \supseteq \{z_1^{k_1}, \cdots z_n^{k_n}\}$.

$\ulcorner$ Suppose we proved the local duality theorem for $\{z_1^{k_1}, \cdots z_n^{k_n}\}$. Suppose $\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|f_i(z)|=\varepsilon} \frac{g(z)\, h(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} = 0$

for all $h(z) \in \mathcal{O}$.

$$\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|f_i(z)|=\epsilon} \frac{g(z)\, h(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)} = \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|z_i|=\epsilon'} \frac{g(z)\, h(z)\, \det A\, dz_1 \wedge dz_n}{z_1^{k_1} \cdots z_n^{k_n}}$$

by the transformation law on P657 ~ P658,
where $z_i^{k_i} = \sum a_{ij}(z)\, f_j(z)$.
$\Rightarrow$ By the assumption for $\{z_1^{k_1}, \cdots z_n^{k_n}\}$, $g \det A \in \{z_1^{k_1}, \cdots z_n^{k_n}\}$.
$\det A$ is unit ? Let's see what is going to happen. $\lrcorner$

Step One. In case $f_i(z) = z_i^{k_i+1}$, we take $h(z) = z_1^{l_1} \cdots z_n^{l_n}$ and write the power series

$$g(z) = \sum g_{i_1, \cdots i_n}\, z_1^{i_1} \cdots z_n^{i_n}.$$

Then, by iterating the usual Cauchy integral formula,

$$\operatorname{res}_f(g, h) = \sum_{i_1, \cdots i_n} g_{i_1 \cdots i_n} \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \int_{|z_i|=\epsilon} \frac{dz_1 \wedge \cdots \wedge dz_n}{z_1^{k_1+1-l_1-i_1} \cdots z_n^{k_n+1-l_n-i_n}}$$

$$= g_{k_1-l_1, \cdots k_n-l_n}.$$

$\ulcorner$ $\operatorname{res}_f(g, h) = \operatorname{Res}_{\{0\}}\left(\frac{g(z)\, h(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)}\right)$