$$- \lambda_2 \frac{dg_2}{g_2} - dk_\beta$$

$$= \lambda_1 \left(\frac{df_1}{f_1} - \frac{dg_1}{g_1} \right) + \lambda_2 \left(\frac{df_2}{f_2} - \frac{dg_2}{g_2} \right) + d(k_\alpha - k_\beta)$$

$$= \lambda_1 \left(\frac{g_1}{f_1} \, d\left(\frac{f_1}{g_1}\right) \right) + \lambda_2 \left(\frac{g_2}{f_2} \, d\left(\frac{f_2}{g_2}\right) \right) + d(k_\alpha - k_\beta)$$

$\Rightarrow$ By ⊛, since f_1/g_1 & f_2/g_2 are non-zero holomorphic functions,

$$\frac{g_1}{f_1} d(f_1/g_1) = d \log\left(\frac{f_1}{g_1}\right).$$

$$\Rightarrow \quad d a^{1,0}_{\alpha\beta} = - a^{0,1}_\beta + a^{0,1}_\alpha = \lambda_1 \, d\log(f_1/g_1) + \lambda_2 \, d\log(f_2/g_2) + d(k_\alpha - k_\beta).$$

$$\Rightarrow \quad a^{1,0}_{\alpha\beta} = \lambda_1 \log(f_1/g_1) + \lambda_2 \log(f_2/g_2) + k_\alpha - k_\beta + C_{\alpha\beta}$$

where $C_{\alpha\beta}$ constant.

$$\begin{aligned}
\Rightarrow \quad (\delta a^{1,0})_{\alpha\beta\gamma} &= a^{1,0}_{\beta\gamma} - a^{1,0}_{\alpha\gamma} + a^{1,0}_{\alpha\beta} \\
&= \lambda_1 \log(f_1/g_1) + \lambda_2 \log(f_2/g_2) + k_\alpha - k_\beta + C_{\alpha\beta} \\
&\quad - (\lambda_1 \log(f_1/h_1) + \lambda_2 \log(f_2/h_2) + k_\alpha - k_\gamma + C_{\alpha\gamma}) \\
&\quad + \lambda_1 \log(g_1/h_1) + \lambda_2 \log(g_2/h_2) + k_\beta - k_\gamma + C_{\beta\gamma}. \\
&= \lambda_1 \, 2\pi\sqrt{-1} \, C_1(D_1) + \lambda_2 \, 2\pi\sqrt{-1} \, C_1(D_2) + C_{\alpha\beta} - C_{\alpha\gamma} + C_{\beta\gamma} \\
&= \lambda_1 \, 2\pi\sqrt{-1} \, C_1(D_1) + \lambda_2 \, 2\pi\sqrt{-1} \, C_1(D_2) + (\delta C)_{\alpha\beta\gamma}
\end{aligned}$$

where $C = (C_{\alpha\beta})$, $(\delta C)_{\alpha\beta\gamma} = C_{\beta\gamma} - C_{\alpha\gamma} + C_{\alpha\beta}$,

$(d+\delta)(C) = \delta C$, $C \in C^1(\underline{U}, \Omega^0(*))$

$$\begin{aligned}
\Rightarrow \quad [\delta a^{1,0}] &= [\lambda_1 \, 2\pi\sqrt{-1} \, C_1(D_1) + \lambda_2 \, 2\pi\sqrt{-1} \, C_1(D_2)] \\
&= \lambda_1 \, 2\pi\sqrt{-1} \, [C_1(D_1)] + \lambda_2 \cdot 2\pi\sqrt{-1} \, [C_1(D_2)].
\end{aligned}$$

Thus

$$\begin{tikzcd}
{[a] = [a^{1,0}+a^{0,1}] \in {}'E_2^{0,1}} \arrow[r] \arrow[d] & {}'E_2^{2,0} \ni [\delta a^{1,0}] = \lambda_1 2\pi\sqrt{-1}\,[C_1(D_1)] + \lambda_2 2\pi\sqrt{-1}\,[C_1(D_2)] \arrow[d] \\
H^1(\underline{U}, \mathcal{O}^*) \arrow[r] & H^2(\underline{U}, \mathbb{C}) \ni [\delta a^{1,0}] = \lambda_1 2\pi\sqrt{-1}\,[C_1(D_1)] + \lambda_2 2\pi\sqrt{-1}\,[C_1(D_2)]
\end{tikzcd}$$

$$\lambda_1 \left(\frac{f_1}{g_1}, \frac{f_1}{h_1}, \frac{g_1}{h_1} \right) + \lambda_2 \left(\frac{f_2}{g_2}, \frac{f_2}{h_2}, \frac{g_2}{h_2} \right) \longrightarrow [\delta a^{1,0}]$$