⌜ Since $\mu_i \in H^{1,1}(S) \cap H^2(S, \mathbb{Z})$, by the Lefschetz (1,1) theorem on P163, $\mu_i = c_1(L_i)$ for a holomorphic line bundle $L_i \longrightarrow S$. ⌟

Since intersection numbers are topological invariant,

$$L_i \cdot L_i = -1, \qquad L_i \cdot L_j = 0 \quad (i \neq j)$$
$$L_i \cdot K_S = E_i \cdot K_{S_0} = -1.$$

⌜ $L_i \cdot L_i \underset{\varphi_t}{=} E_i \cdot E_i = -1.$ $\qquad L_i \cdot L_j \underset{\varphi_t}{=} E_i \cdot E_j = 0$

$L_i \cdot K_S \overset{?}{=}$

$$\begin{tikzcd}
i^*[H] \arrow[r] \arrow[d] & {[H]} \arrow[d] \\
\widetilde{\mathbb{P}}^2 \arrow[r, "i"] & \mathbb{P}^3 \\
 & S \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Recall that $i(E_i)$ is a line in $\mathbb{P}^3$, let $i(E_i) = \ell$.

$$\begin{aligned}
L_i \cdot K_S &= -L_i \cdot (H \cap S) \\
&= -L_i \cdot (H \cap S) \\
&= -\#(i(E_i) \cap (H \cap S_0)), \ (\text{let } \ell = i(E_i)) \\
&= -\#\ell \cap H = -1 = L_i \cdot K_S
\end{aligned}$$ ⌟

φ_t is a "homotopy".

"We calculate the intersection # for S_0, and through φ_t for S."

Now, as remarked, $h^{2,0}(S) = 0$; by the Lefschetz hyperplane theorem

$$H^1(S, \mathbb{Q}) \cong H^1(\mathbb{P}^3, \mathbb{Q}) = 0,$$

so $h^{1,0}(S) = 0$ and consequently $\chi(\mathcal{O}_S) = 1$.

⌜ Since $[S] \longrightarrow \mathbb{P}^3$ is positive ($\because$ $S \sim 3H$ and $[H] \to \mathbb{P}^3$ is positive), by the Lefschetz hyperplane theorem on P156,

$H^1(\mathbb{P}^3, \mathbb{Q}) \cong H^1(S, \mathbb{Q})$, $1 \leq 3-2$. $\Rightarrow$ $H^1(\mathbb{P}^3, \mathbb{Q}) = H^1(\mathbb{P}^3, \mathbb{C}) = 0$ $\Rightarrow$ $H^1(S, \mathbb{Q}) = H^1(S, \mathbb{C}) = 0$. $\Rightarrow$ $H^1(S) = H^{1,0}(S) \oplus H^{0,1}(S)$ $\Rightarrow$ $H^{1,0}(S) = 0$ $\Rightarrow$ $\chi(\mathcal{O}_S) = h^{0,0}(S) + (-1)h^{0,1}(S) + h^{0,2}(S) = h^{0,0}(S) = 1$ ⌟