Cohomology of Sheaves

$\mathcal{F}$ sheaf on M. $\underline{\mathcal{U}} = \{U_\alpha\}$ locally finite open cover of M.

Define
$$C^0(\underline{\mathcal{U}}, \mathcal{F}) = \prod_\alpha \mathcal{F}(U_\alpha)$$
$$C^1(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha \neq \beta} \mathcal{F}(U_\alpha \cap U_\beta)$$
$$\vdots$$
$$C^p(\underline{\mathcal{U}}, \mathcal{F}) = \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_p} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}) .$$

An element $\sigma = \{\sigma_I \in \mathcal{F}(\cap U_{i_k})\}_{\# I = p+1}$ of $C^p(\underline{\mathcal{U}}, \mathcal{F})$ is called a p-cochain of $\mathcal{F}$.
Define a coboundary operator

$\delta : C^p(\underline{\mathcal{U}}, \mathcal{F}) \longrightarrow C^{p+1}(\underline{\mathcal{U}}, \mathcal{F})$ by the formula
$$(\delta\sigma)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \sigma_{i_0 \cdots \hat{i_j} \cdots i_{p+1}} |_{U_{i_0} \cap \cdots U_{i_{p+1}}}$$

In particular, if $\sigma = \{\sigma_u\} \in C^0(\underline{\mathcal{U}}, \mathcal{F})$,
$$(\delta\sigma)_{u,v} = \sum_{j=0}^{1} (-1)^j \sigma_{\hat{i_0} \hat{i_1}} |_{u \cap v}$$
$$= -\sigma_u |_{u \cap v} + \sigma_v |_{u \cap v} = \sigma_v - \sigma_u$$ and if
$\sigma = \sigma_{u,v} \in C^1(\underline{\mathcal{U}}, \mathcal{F})$.
$$\cancel{\delta\delta}\ (\delta\sigma)_{uvw} = \sigma_{vw} - \sigma_{uw} + \sigma_{uv}$$ (omitting the restriction)

A p-cochain $\sigma \in C^p(\underline{\mathcal{U}}, \mathcal{F})$ is called a cocycle if $\delta\sigma = 0$.