$$\check{H}^*(M.\ \mathbb{Z}) \cong H^*(\underline{U}\ \mathbb{Z}) \cong H^*(K.\ \mathbb{Z}).$$

The de Rham Theorem.

M real C^∞– manifold.

We say that a singular p-chain σ on M, given as a formal linear combination $\Sigma a_i f_i$ of maps $\Delta \xrightarrow{f_i} M$ of the standard p-simplex $\Delta \subset \mathbb{R}^p$ to M, is piecewise smooth if the maps f_i extend to C^∞ maps of a nbd of Δ to M.

$C_p^{PS}(M.\ \mathbb{Z})$ = Space of piecewise smooth integral p-chains

Clearly the boundary of a piecewise smooth chain is again piecewise smooth, so that $C_*^{PS}(M.\ \mathbb{Z})$ forms a subcomplex of $C_*(M, \mathbb{Z})$.

Set $Z_p^{PS}(M.\ \mathbb{Z}) = \ker \partial :\ C_p^{PS}(M.\ \mathbb{Z}) \longrightarrow C_{p-1}^{PS}(M.\ \mathbb{Z})$

$$H_p^{PS}(M.\ \mathbb{Z}) = \frac{Z_p^{PS}(M.\ \mathbb{Z})}{\partial C_{p+1}^{PS}(M.\ \mathbb{Z})}.$$

$\Rightarrow$ By a foundational result from differential topology, the inclusion map $C_*^{PS}(M.\ \mathbb{Z}) \longrightarrow C_*(M.\ \mathbb{Z})$ induces an isomorphism

$$H_p^{PS}(M.\ \mathbb{Z}) \cong H_p(M.\ \mathbb{Z}).$$ (See Massey p 253. Singular H.T. Th 2.1)

in other words, every homology class in $H_p(M.\ \mathbb{Z})$ can be represented by a piecewise smooth p-cycle, and if a piecewise smooth p-cycle σ is homologous to 0 in the usual sence, $\exists$ a piecewise smooth $(p+1)$-chain τ with $\partial\tau = \sigma$.