We now come to the Künneth formular. Given compact, complex manifolds M and N, we consider the product $M \times N$. The projections onto the two factors induce maps

$$H^*(M, \Omega^p_M) \longrightarrow H^*(M \times N, \Omega^p_{M\times N}),$$

$$H^*(N, \Omega^p_N) \longrightarrow H^*(M \times N, \Omega^p_{M\times N}).$$

We will prove in a minute that these are injective, and will identify these groups with their images. This being understood, the cup product gives

$$(*) \quad H^*(M, \Omega^*_M) \otimes H^*(M, \Omega^*_N) \longrightarrow H^*(M\times N, \Omega^*_{M\times N}).$$

The Künneth formula asserts that this is an isomorphism.

We will prove this using harmonic forms. Hermitian metrics on M and N induce the product metric on $M\times N$, and we will show that, with this choice of metrics,

$$(**), \quad \mathcal{H}^{u,v}(M\times N) \cong \bigoplus_{\substack{p+r=u \\ q+s=v}} (\mathcal{H}^{p,q}(M) \otimes \mathcal{H}^{r,s}(N))$$

This will establish the Künneth theorem.

To carry this out, we denote by z, w generic local coordinates on M and N. Given forms φ, η on M, N, respectively, we will denote by $\varphi\otimes\eta$ the induced form on $M\times N$ given by

$$\varphi\otimes\eta\,(z, w) = \varphi(z)\wedge\eta(w).$$

These forms will be said to be decomposable.