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($\Leftarrow$) Suppose $\sigma_1, \sigma_2, \cdots \sigma_{k-r+1}$ are linearly dependent at x.
Assume $\sigma_1, \cdots \sigma_{k-r}$ are linearly independent at x, and $\sigma_1, \cdots \sigma_{k-r}, \omega_{k-r+1}$ are linearly independent at x.

$\Rightarrow$

$$
\begin{array}{lllllll}
 & & & & k-r & k-r+1 & k-r+2 \\
\sigma_1 = & \sigma_1 + 0\,\sigma_2 + \cdots & & & + 0 & + 0 & + 0 \\
\sigma_2 = & 0 + & \sigma_2 + 0 & & + 0 & + 0 & + 0 \\
\sigma_3 = & 0 + 0 & + \sigma_3 & & 0 & + 0 & + 0 \\
\vdots & & & & \vdots & \vdots & \vdots \\
\sigma_{k-r} = & 0 + & 0 + & \cdots & + \sigma_{k-r} & + 0 & + 0 \\
\sigma_{k-r+1} = & *\sigma_1 + *\sigma_2 & + \cdots & & + *\sigma_{k-r} & + 0 & + 0 \\
\sigma_{k-r+2} = & *\sigma_1 + *\sigma_2 & + \cdots & & + *\sigma_{k-r} & + *\omega_{k-r+1} & + * \\
\vdots & & & & & &
\end{array}
$$

$$
\begin{array}{ll}
 & \qquad\qquad\qquad\quad k-r+1 \\
\Rightarrow & (1, 0, \cdots 0, *, * \cdots *) \\
 & (0, 1, \cdots 0\ * \cdots * \quad *) \\
 & \vdots \\
k-r \Rightarrow & (0, 0 \cdots 1, * \cdots * \quad *) \\
k-r+1 \Rightarrow & (0\ 0 \cdots 0\ 0\ * \cdots *) \\
k-r+2 \Rightarrow & (0, 0 \cdots 0\ 0\ * \cdots *)
\end{array}
$$

$\Rightarrow$ $L(x) = \Lambda + V_{k+r+n-1}$ does not span $\mathbb{C}^n$.

$\Rightarrow$ By $\dim(\Lambda \cap V_{n-k+r-1}) = n + r - 1 - \dim(\Lambda + V_{n-k+r-1})$, $\dim(\Lambda \cap V_{n-k+r-1}) \geq r$.

In the assumption above, we can choose r' arbitrarily, so if $\sigma_1, \cdots \sigma_{k-r'+1}$ are lin-dependent at x and $\sigma_1, \cdots \sigma_{k-r'}$ are linearly independent at x, by the argument above, we can conclude that $\dim(\Lambda + V_{n-k+r'-1}) \leq n-1$.

$\Rightarrow$ If $r \geq r'$, since $V_{n-k+r'-1} \supset V_{n-k+r-1}$,
$\dim(\Lambda + V_{n-k+r-1}) \leq n-1$.