$U_\alpha \times \mathbb{C}$ over all α and identifying $\{z\} \times \mathbb{C}$ in $U_\alpha \times \mathbb{C}$ and $U_\beta \times \mathbb{C}$ via multiplication by $g_{\alpha\beta}(z)$.

Now, given L as above, for any collection of nonzero holomorphic functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ we can define alternate trivializations of L over $\{U_\alpha\}$ by

$$\varphi_\alpha' = f_\alpha \cdot \varphi_\alpha :$$

transition functions $g'_{\alpha\beta}$ for L relative to $\{\varphi_\alpha'\}$ will then be given by

$$(**) \qquad g'_{\alpha\beta} = \frac{f_\alpha}{f_\beta} \cdot g_{\alpha\beta} .$$

$\ulcorner$ $\quad g'_{\alpha\beta}(z) = (\varphi_\alpha' \circ \varphi_\beta'^{-1})|_{L_z} = (f_\alpha \cdot \varphi_\alpha \circ (f_\beta \cdot \varphi_\beta)^{-1})|_{L_z}$

$= (f_\alpha \cdot \varphi_\alpha (f_\beta^{-1} \varphi_\beta^{-1}))|_{L_z} = (f_\alpha \cdot f_\beta^{-1} \cdot \varphi_\alpha \circ \varphi_\beta^{-1})|_{L_z} = f_\alpha \cdot f_\beta^{-1} (\varphi_\alpha \circ \varphi_\beta^{-1})|_{L_z}$

$= \dfrac{f_\alpha(z)}{f_\beta(z)} (\varphi_\alpha \circ \varphi_\beta^{-1})|_{L_z} \quad \Longrightarrow \quad g'_{\alpha\beta} = \dfrac{f_\alpha}{f_\beta} g_{\alpha\beta}$ $\lrcorner$

On the other hand, any other trivialization of L over $\{U_\alpha\}$ can be obtained in this way, and so we see that collections $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ of transition function define the same line bundle $\Longleftrightarrow$ $\exists$ functions $f_\alpha \in \mathcal{O}^*(U_\alpha)$ satisfying $(**)$.

The description of line bundles by transition functions lends itself well to a sheaf-theoretic interpretation. First, the transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ for a line bundle $L \longrightarrow M$ represent a Čech 1-cochain on M with coefficients in $\mathcal{O}^*$; the relation simply asserts that $\delta(\{g_{\alpha\beta}\}) = 0$, i.e. $\{g_{\alpha\beta}\}$ is a Čech cocycle. Moreover, by the last paragraph, two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ define the same line bundle $\Longleftrightarrow$ their difference $\{g_{\alpha\beta} \cdot g'^{-1}_{\alpha\beta}\}$ is a Čech coboundary; consequently the set of line bundles on M is just $H^1(M, \mathcal{O}^*)$.