The reader may verify that the birational map ψ is given by the linear system of quintic curves in $\mathbb{P}^2$ having double points at each of the points a_i. Of course, blowing down any of the 72 sets of six disjoint lines on the cubic $\tilde{\mathbb{P}}^2$ yields a Cremona transformation.

Γ

$$\begin{tikzcd} & \tilde{\mathbb{P}}^2_{a_1 \cdots a_6} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{\pi}"] & \\ \mathbb{P}^2 \arrow[rr, "\psi"] & & \mathbb{P}^2 \end{tikzcd}$$

$x \longmapsto [\sigma_0(x), \sigma_1(x), \sigma_2(x)]$, where by the exact correspondence on P492, $\langle \sigma_0, \sigma_1, \sigma_2 \rangle$ is a linear system on $\mathbb{P}^2$ with base locus of codimension 2.

Let $\psi(G_i) = C_i$.

$\Rightarrow$ By the arguments and results on P489, no three of C_i's are collinear. $\Rightarrow \{C_1, C_2, C_3\}$ spans $\mathbb{P}^2$. $\Rightarrow \exists$ linearly independent $b_1, b_2, b_3 \in \mathbb{C}^3$ s.t.

$b_1 = (b_{11}, b_{12}, b_{13})$ $\quad b_2 = (b_{21}, b_{22}, b_{23})$ $\quad b_3 = (b_{31}, b_{32}, b_{33})$

$$b_{11}\sigma_0 + b_{12}\sigma_1 + b_{13}\sigma_2 = \eta_1 \qquad \{\eta_1 = 0\} \ni C_2, C_3.$$
$$b_{21}\sigma_0 + b_{22}\sigma_1 + b_{23}\sigma_2 = \eta_2 \qquad \{\eta_2 = 0\} \ni C_1, C_3$$
$$b_{31}\sigma_0 + b_{32}\sigma_1 + b_{33}\sigma_2 = \eta_3 \qquad \{\eta_3 = 0\} \ni C_1, C_2.$$

$\Rightarrow$ η_1 is divisible by G_2, G_3 $\quad\Rightarrow\quad \eta_1 = G_2 G_3 l_1$

η_2 " by G_1, G_3 $\quad \eta_2 = G_1 G_3 l_2$

η_3 " by G_1, G_2 $\quad \eta_3 = G_1 G_2 l_3$