Date

ension and codimension of cycles, but when Grassmannians arise in geometric questions we will generally want to think of them in the latter way.

The Cell Decomposition.

Recall that the cell decomposition

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^1 \cup \mathbb{C}^0$$

of $\mathbb{P}^n = G(1, n+1)$ is obtained by choosing a flag

$$V = (V_1 \subsetneqq \cdots \subsetneqq V_{n-1} \subsetneqq V_n \subsetneqq \mathbb{C}^{n+1})$$

of linear subspaces of $\mathbb{C}^{n+1}$ and taking $W_i \cong \mathbb{C}^{i-1} = \{ l \subset \mathbb{C}^{n+1} : l \subset V_i, \ l \not\subset V_{i-1} \}$. 「 See P60. 」

The same technique works to give a cell decomposition of the Grassmannian: if we set $V_i = \{e_1, \cdots e_i\} \subset \mathbb{C}^n$, then the set of $\Lambda \in G(k, n)$ whose intersection with each V_i is of a specified dimension turns out, as we shall see, to be a simple cell. The set-up is as follows: for every $\Lambda \in G(k, n)$ consider the increasing sequence of subspaces

$$(*) \quad 0 \subset \Lambda \cap V_1 \subset \Lambda \cap V_2 \subset \cdots \subset \Lambda \cap V_{n-1} \subset \Lambda \cap V_n = \Lambda.$$

For generic Λ, $\underset{i > n-k}{\Lambda \cap V_i}$ will be zero for $i \leq n-k$, and $(i+k-n)$-dimensional <u>thereafter</u> — indeed, we have seen that the set of such Λ is just the open set $U_{I^\circ} \cong \mathbb{C}^{k(n-k)} \subset G(k, n)$. Now, for any sequence of integers $a_1, a_2 \cdots a_k$, set

$$W_{a_1, \cdots a_k} = \{ \Lambda \in G(k, n) : \dim (\Lambda \cap V_{n-k+i-a_i}) = i \}.$$

「 This is just a definition. Thereafter means that $i > n-k$. $\dim \Lambda = k$. $\Lambda \cap V_i = k+i-n$ for generic Λ, i.e. $\Lambda + V_i = \mathbb{C}^n$. 」

We observe that $\dim (\Lambda + V_{n-k+i-a_i}) = n - a_i$, and conse-

PAGE