Tor and the Syzygy Theorem. Having used Ext to put the local duality theorem in final form, we shall use Tor to prove the

Syzygy Theorem. Let M be a finitely generated $\mathcal{O}$-module and

$$0 \to F \to E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$$

an exact sequence of $\mathcal{O}$-modules where the E_k are projective (= free). Then F is also projective.

Let $m = (z_1, \cdots z_n)$ be the maximal ideal and $\mathbb{C} = \mathcal{O}/m$ considered as an $\mathcal{O}$-module. We begin by proving:

Lemma. $\mathrm{Tor}_1^{\mathcal{O}}(\mathbb{C}, N) = 0 \Rightarrow N$ is free $\mathcal{O}$-module.

Proof. We first remark that

$$\mathrm{Tor}_0^{\mathcal{O}}(M, N) \cong M \otimes_{\mathcal{O}} N.$$

$\ulcorner$ $\mathrm{Tor}_0^{\mathcal{O}}(M, N) = H_0(E.(M) \otimes_{\mathcal{O}} N)$

$$E_1 \otimes_{\mathcal{O}} N \longrightarrow E_0 \otimes_{\mathcal{O}} N \longrightarrow M \otimes_{\mathcal{O}} N \longrightarrow 0$$

$$H_0(E.(M) \otimes_{\mathcal{O}} N) = \frac{E_0 \otimes_{\mathcal{O}} N}{(\partial \otimes id)(E_1 \otimes_{\mathcal{O}} N)} \cong \frac{E_0 \otimes N}{\partial E_1 \otimes N} \cong E_0/\partial E_1 \otimes N$$

$\cong M \otimes_{\mathcal{O}} N$. See note p590. $\lrcorner$

This is because $\otimes$ is right-exact, so that $E_1 \to E_0 \to M$