Intuitively, this amounts to saying that the analytic varieties $V_i = \{f_1(z) = \cdots = f_i(z) = 0\}$ have codimension exactly equal to i, which seems quite reasonable and may be rigorously proved in the case $n=2$ as follows:

We need to show that if $g f_2 = 0$ in $\mathcal{O}/\{f_1\}$, then $g = hf_1$ is a multiple of f_1. The problem is unchanged if we multiply the f_i's or g by units, and so we may choose coordinates (z, w) such that $f_1, f_2, g \in \mathcal{O}_z[w]$ are all Weierstrass polynomials. By the division theorem,

$$g = hf_1 + r,$$

where $r \in \mathcal{O}_z[w]$ is a polynomial of degree less than $d = \deg f_1$. ☞ See P11. Weierstrass division theorem

For $|z| \le \varepsilon$, we denote by $w_1(z), \cdots w_d(z)$ the roots of

$$f_1(z, w) = 0$$

where some roots may be repeated. Then, since the equations $f_1(z, w) = f_2(z, w) = 0$ have only $(0, 0)$ as common solution, for z^* close to zero all $f_2(z^*, w_\nu(z^*)) \neq 0$. But then, since by assumption $g(z, w_\nu(z)) f_2(z, w_\nu(z)) = 0$, the equation $r(z^*, w) = 0$ will have $d > \deg r$ roots. Hence $r \equiv 0$ and $g = hf_1$. Q. E. D.

☞ $g(z, w_\nu(z)) f(z, w_\nu(z)) = 0$ & $f_2(z, w_\nu(z)) = 0$ for sufficiently small $z \neq 0$, $\Rightarrow$ $g(z, w_\nu(z)) = 0 = h(z, w_\nu(z)) f_1(z, w_\nu(z)) + r(z, w_\nu(z)) = r(z, w_\nu(z))$ for all $(z, w_\nu(z))$
$\Rightarrow$ $r = 0$ have d roots $(z^*, w_\nu(z^*))$, $\nu = 1, \cdots d$.

If $w_1(z^*) = w_2(z^*)$, $\frac{\partial g}{\partial w} = \frac{\partial h}{\partial w} f_1 + h \frac{\partial f_1}{\partial w} + \frac{\partial r}{\partial w}$ & plug in $w_1(z^*)$,