Cohomology of Currents

On a manifold M we have defined the complex of currents $(\mathcal{D}^*(M), d)$. The inclusion of smooth forms into the currents gives a natural map

$$H^*_{DR}(M) \longrightarrow H^*(\mathcal{D}^*(M), d)$$

from de Rham cohomology into the cohomology computed from currents.

⸢ See P369, $d : \mathcal{D}^q(M) \longrightarrow \mathcal{D}^{q+1}(M)$ defined by

$$(dT)(\varphi) = (-1)^{q+1} T(d\varphi), \quad \varphi \in A^{n-q-1}(M).$$

Given $\omega \in A^q(M)$, define $T_\omega \in \mathcal{D}^q(M)$ by

$$T_\omega(\varphi) = \omega \wedge \varphi \quad , \quad \varphi \in A^{n-q}(M).$$

$$\Rightarrow \quad \begin{array}{ccc} A^q(M) & \longrightarrow & \mathcal{D}^q(M) \\ \psi & & \psi \\ \omega & \longmapsto & T_\omega . \end{array} \quad \text{is injective since}$$

$T_\omega(*\omega) = 0 \Rightarrow \omega = 0.$

$$\begin{array}{ccccc} \mathcal{D}^q(M) & \longrightarrow & \mathcal{D}^{q+1}(M) & \longrightarrow & \mathcal{D}^{q+2}(M) \\ \uparrow \ \ni T_\omega & \longmapsto & dT_\omega \ \uparrow & & \uparrow \\ A^q(M) & \longrightarrow & A^{q+1}(M) & \longrightarrow & A^{q+2}(M) \\ \psi & & \psi & & \\ \omega & \longmapsto & d\omega & & \end{array}$$

$(dT_\omega)(\varphi) = (-1)^{q+1} T_\omega(d\varphi) = (-1)^{q+1} \omega \wedge d\varphi = -d(\omega\wedge\varphi)$
$+ d\omega \wedge \varphi = -d(\omega\wedge\varphi) + T_{d\omega}(\varphi) = (-1)^{q+1} T_\omega(d\varphi)$
$\Rightarrow (dT_\omega - T_{d\omega})(\varphi) = -d(\omega\wedge\varphi). \Rightarrow$