It follows that for $q \in V''$ near p. $V = V''$ is a manifold in a nbd of q and so $V_s \subset (|(\partial f_i/\partial z_j)_{1 \le i, j \le k}| = 0)$. ⌟

It is in fact the case that V_s is an analytic subvariety of M — if we choose local defining functions $f_1 \cdots f_k$ for V carefully, V_s will be the common zero locus of the determinants of the $k \times k$ minors of $\mathcal{J}(f)$. For our purposes, however, we simply need to know that the singular locus of an analytic variety is comparatively small, and so we will not prove this stronger assertion.

We state one more result on analytic varieties:

Proposition. An analytic variety V is irreducible if and only if V^* is connected.

Proof. One direction is clear: if $V = V_1 \cup V_2$ with $V_1, V_2 \subsetneq V$ analytic varieties, then $(V_1 \cap V_2) \subset V_s$, so V^* is disconnected.

$\boxed{V_1 \cap V_2 \subset V_s, \quad V_1^* \subset V_1 \quad V_2^* \subset V_2}$ *

$$V^* \cap (V_1 \cap V_2) = \phi \quad \text{since} \quad V_1 \cap V_2 \subset V_s.$$
$$\Rightarrow (V^* \cap V_1) \cup (V^* \cap V_2) = V^* \cap (V_1 \cup V_2) = V^*$$
$$\text{and} \quad (V^* \cap V_1) \cap (V^* \cap V_2) = V^* \cap (V_1 \cap V_2) = \phi$$
$$\Rightarrow V^* = (V^* \cap V_1) \amalg (V^* \cap V_2) \quad \text{disjoint union.}$$ ⌟