Now $d(|S|^2) = d(\lambda \cdot \bar{\lambda} h)$
$$= h\bar{\lambda}\, d\lambda + h\lambda\, d\bar{\lambda} + |\lambda|^2 dh .$$

So we have $\theta + \bar{\theta} = \dfrac{dh}{h}$,

i.e., $\theta = \partial \log h = \partial \log |e|^2$, and
$$\Theta = d\theta - \theta\wedge\theta = d\theta = \bar{\partial}\partial \log |e|^2$$
$$= 2\pi i\, dd^c \log |e|^2 .$$

$\ulcorner$ θ is type of $(1,0)$

$$\theta + \bar{\theta} = \frac{\partial h}{h} + \frac{\bar{\partial} h}{h} \Rightarrow \theta = \frac{\partial h}{h} = \partial \log h .$$

$$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial) \qquad d = \bar{\partial} + \partial$$

$$\Rightarrow d^c d = \frac{i}{4\pi}(\bar{\partial}-\partial)(\bar{\partial}+\partial) = \frac{i}{4\pi}(-\partial\bar{\partial} + \bar{\partial}\partial)$$
$$= \frac{i}{4\pi}(-\partial\bar{\partial} - \partial\bar{\partial}) = -\frac{i}{2\pi}\partial\bar{\partial}$$

$$\Rightarrow dd^c = \frac{i}{2\pi}\partial\bar{\partial} \qquad \Rightarrow \partial\bar{\partial} = -2\pi i\, dd^c = -\bar{\partial}\partial$$ ⌟

Note that this holds for any nonzero holomorphic section e.

Now let $D = V$ be given by local data f_α and let s be a global section $\{f_\alpha\}$ of D vanishing exactly on V. Set

$$D(\epsilon) = (\,|s(z)| < \epsilon\,) \subset M.$$

$\ulcorner$ $[D] = L \Rightarrow$ local data f_α give a global section $s := \{f_\alpha\}$ of D. ⌟