criminant of $P(A, z)$ is zero, the set of A's in GL_n whose $P(A, z)$ has a double root is a subvariety of GL_n. $\Rightarrow$ $\mathcal{A}$ is dense.

Clearly $\mathcal{A}$ is in $\mathcal{K}$ which is path-connected.

$$f(A) = G(P'(A), \cdots P^n(A)) \text{ for all } A \in \mathcal{K}.$$

Thus since GL_n is open (in M_n) and connected,

$$f(A) = G(P'(A), \cdots P^n(A)) \text{ for all } A \in M_n.$$

"Comment on $\pi_0(GL_n) = 0$"

See P 55.7 of Lecture notes on K-theory.

Now, a k-linear form

$$\tilde{P} : M_n \times \cdots \times M_n \longrightarrow \mathbb{C}$$

is called invariant if for any $A_1, \cdots A_k \in M_n$, $g \in GL_n$,

$$\tilde{P}(A_1, \cdots A_k) = \tilde{P}(gA_1g^{-1}, \cdots gA_kg^{-1}).$$

An invariant form $\tilde{P}$ clearly gives an invariant polynomial P by

$$P(A) = \tilde{P}(A, \cdots A).$$

$$P(gAg^{-1}) = P(A) \qquad P\left(a_{11}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + a_{12}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \cdots\right)$$

$$= \tilde{P}(\cdots \quad) = a_{11} \cdots a_{11}\, \tilde{P}\left(\begin{pmatrix}1 & \\ & 0\end{pmatrix}, \begin{pmatrix}1 & \\ & 0\end{pmatrix} \cdots\right) + \cdots$$

which is a polynomial in A

In fact, the converse is also true: any invariant polynomial P of degree k can be realized as the restriction of a symmetric invariant k-linear form $\tilde{P}$ on $M_n \times \cdots \times M_n$ to the diagonal. The form $\tilde{P}$, called the polarization of P, is uniquely determined