Kodaira – Serre Duality Theorem

1. $H^n(M.\Omega^n) \xrightarrow{\simeq} \mathbb{C}$ and
2. the pairing

$$H^q(M.\Omega^p) \otimes H^{n-q}(M.\Omega^{n-p}) \longrightarrow H^n(M.\Omega^n)$$

is nondegenerate.

pf) The mapping in 1 is given by composing

$$H^n(M.\Omega^n) \cong H_{\bar\partial}^{n.n}(M) \quad \text{with the linear function}$$

$$H_{\bar\partial}^{n.n}(M) \longrightarrow \mathbb{C} \qquad \text{defined by} \quad \psi \longmapsto \int_M \psi$$

which is well-defined on account of Stokes' theorem and $d = \bar\partial$ on $A^{n.n-1}(M)$. The fact that 1 is an isomorphism results from

$$H_{\bar\partial}^{n.n}(M) \cong \mathcal{H}^{n.n}(M) \cong \mathbb{C}\,\Phi,$$

since $\int_M \Phi = \mathrm{Vol}(M) > 0$.

The pairing 2. is given by composing

$$H^q(M.\Omega^p) \cong H_{\bar\partial}^{p.q}(M) \quad \text{with the pairing}$$

$$H_{\bar\partial}^{p.q}(M) \otimes H_{\bar\partial}^{n-p.n-q}(M) \longrightarrow \mathbb{C}$$

defined by

$$\psi \otimes \eta \longmapsto \int_M \psi \wedge \eta .$$

It is nondegenerate, since

$$H_{\bar\partial}^{p.q}(M) \cong \mathcal{H}^{p.q}(M),$$

and for a harmonic form $\psi \neq 0$,

$$\psi \otimes *\psi \longmapsto \int_M \psi \wedge *\psi = \|\psi\|^2 > 0 \qquad \text{Q.E.D.}$$