$$= \qquad \sum \alpha(w_\nu) - \frac{1}{2\pi\sqrt{-1}} \int_{\partial\Delta} \frac{\alpha(w)}{w - w_\nu} d\bar{w}$$

If we choose $\Delta \supset \operatorname{supp}\alpha$, then $\alpha = 0$, on $\partial\Delta$.
$\Rightarrow$ We have what we wanted. ⌟

Returning to the general Poincare-Lelong formula, we must show that $\log|f|$ is integrable and

$$\frac{\sqrt{-1}}{\pi} \int_M \log|f|\, \bar{\partial}\partial\varphi = \int_{Z^*} \varphi,$$

for $\varphi \in A_c^{n-1,\, n-1}(M)$.

⌜ $p(w) \in \mathcal{O}(\mathbb{C}) \qquad Z = \{w_\nu\}$.

$$T_Z = \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log|p(w)| \Longleftrightarrow \quad T_{\{w_\nu\}}(\alpha) = \int_{\{w_\nu\}} \alpha = \sum \alpha(w_\nu)$$

$$= \qquad \left(\frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log|p(w)|\right)(\alpha)$$

$$= - \int_{\mathbb{C}} \frac{\sqrt{-1}}{\pi} \partial \log|p(w)| \wedge \frac{\partial\alpha}{\partial\bar{w}} d\bar{w} = \frac{1}{2\pi\sqrt{-1}} \int_{\mathbb{C}} \partial \log p(w) \wedge \frac{\partial\alpha}{\partial\bar{w}} d\bar{w}$$

$$= \left(\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log|p(w)|^2\right)(\alpha)$$

$$= \frac{\sqrt{-1}}{2\pi}(-1)\int \log|p(w)|^2 \wedge (\bar{\partial}\partial\alpha) = \frac{+1}{2\pi\sqrt{-1}} \int \partial \log|p(w)|^2 \wedge \bar{\partial}\alpha$$

$$= \frac{+1}{2\pi\sqrt{-1}} \int \partial \log p \wedge \bar{\partial}\alpha$$

⌟

The problem is local around a point $p \in Z$ with local