Clearly, rank $Q = n+1 \iff \{QX=0\} = \{0\} \iff$ F is smooth. General case is clear. $\lrcorner$

Indeed, we can be more explicit in our description: suppose $F \subset \mathbb{P}^n$ is a quadric of rank k with singular set $\Lambda \cong \mathbb{P}^{n-k}$, and take V_{k-1} a generic $(k-1)$-plane complementary to, i.e., disjoint from, Λ; $\tilde{F} = F \cap V_{k-1}$ is then a smooth quadric of dimension $k-2$.

$\ulcorner$ $\dim \Lambda + \dim V_{k-1} = n-1 < n$.

$\Rightarrow$ A generic $(k-1)$-plane is disjoint from Λ.

$\Rightarrow$ $\tilde{F} = F \cap V_{k-1}$ is a smooth quadric as we shall see.

By changing the coordinates, we may assume $V_{k-1} = \{X_k = \cdots = X_n = 0\}$, and $\Rightarrow$ $\tilde{F} = F \cap V_{k-1}$ is quadric, and $\dfrac{\partial Q}{\partial X_i} = 2 \sum q_{ij} X_j \Rightarrow$ $\Lambda = \{X_0 = X_1 = \cdots = X_{k-1} = 0\}$.

$$\frac{\partial Q}{\partial X_i} = 2 \sum_{j=0}^{k-1} q_{ij} X_j \quad \text{on } V_{k-1} \ \& \ \frac{\partial Q}{\partial X_i} = 0 \text{ on } \Lambda$$

$$\Rightarrow \sum_{j \geq k}^{n} q_{ij} X_j = 0 \Rightarrow q_{ij} = 0 \text{ if } i \geq k \text{ or } j \geq k$$

$$\Rightarrow Q = \begin{pmatrix} \ast & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \left(\left.\frac{\partial Q}{\partial X_1}\right|_{V_{k-1}} \cdots \left.\frac{\partial Q}{\partial X_{k-1}}\right|_{V_{k-1}} \right) \neq 0 \quad \text{since } \nabla Q \neq 0 \text{ on } V_{k-1}.$$

$\Rightarrow$ $\tilde{F} = F \cap V_{k-1}$ is smooth. $\lrcorner$

Now if L is any line in $\mathbb{P}^n$ meeting both Λ and $\tilde{F}$, L meets F three times and and so is contained in F.