de Rham's theorem.

$$\begin{array}{ccccc} C^p(\underline{U},\mathcal{F}) & \otimes & C^q(\underline{U},\mathcal{G}) & \xrightarrow{\ f\ } & C^{p+q}(\underline{U},\mathcal{H}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \alpha & \otimes & \beta & \longmapsto & f(\alpha\otimes\beta) \end{array}$$

$f : \mathcal{F}\otimes\mathcal{G} \longrightarrow \mathcal{H}$ morphism.

$$f(\alpha\otimes\beta)_{i_0 i_1\cdots i_{p+q}} = f(\alpha_{\lambda_p}\otimes\beta_{\tau_{n-p}})$$

where $\alpha_{\lambda_p} = \alpha_{i_0, i_1,\cdots i_p}$, $\beta_{\tau_{n-p}} = \beta_{i_p i_{p+1}\cdots i_{p+q}}$

Define a coboundary operator on $C^p(\underline{U},\mathcal{F})\otimes C^q(\underline{U},\mathcal{G})$ by

$$\delta(\alpha\otimes\beta) = \delta\alpha\otimes\beta + (-1)^p\alpha\otimes\delta\beta.$$

Prove the following relation.

$$\delta f(\alpha\otimes\beta) = f(\delta\alpha\otimes\beta) + (-1)^p f(\alpha\otimes\delta\beta).$$

LHS. $\delta(f(\alpha\otimes\beta))_{i_0\cdots i_{p+q+1}}$

$$= \sum_{k=0}^{p+q+1}(-1)^k f(\alpha\otimes\beta)_{i_0\cdots\hat{i}_k\cdots i_{p+q+1}} = \sum_{k\le p}(-1)^k f(\alpha\otimes\beta)_{i_0\cdots\hat{i}_k\, i_p\cdots i_{p+q+1}}$$

$$+ \sum_{k>p}(-1)^k f(\alpha\otimes\beta)_{i_0\cdots i_k\, i_p\cdots\hat{i}_k\cdots i_{p+q+1}}$$

$$= \sum_{k\le p}(-1)^k f(\alpha_{i_0\cdots\hat{i}_k\, i_{p+1}}\otimes\beta_{i_{p+1}\cdots i_{p+q+1}}) + \sum_{k\ge p+1}(-1)^k f(\alpha_{i_0\cdots i_p}\otimes\beta_{i_p\cdots\hat{i}_k\cdots i_{p+q+1}})$$

$$= \sum_{k\le p+1} f((-1)^k\alpha_{i_0\cdots\hat{i}_k\, i_{p+1}}\otimes\beta_{i_{p+1}\cdots i_{p+q+1}}) + \sum_{k\ge p+2}(-1)^k f(\alpha_{i_0\cdots i_p}\otimes\beta_{i_p\cdots\hat{i}_k\cdots i_{p+q+1}})$$

$$+ f((-1)^{p+1}\alpha_{i_0\cdots i_p\hat{i}_{p+1}}\otimes\beta_{i_p i_{p+2}\cdots i_{p+q+1}})$$

$$+ f((-1)^{p+1}\alpha_{i_0\cdots i_p\hat{i}_{p+1}}\otimes\beta_{i_{p+1}\cdots i_{p+q+1}}) - f((-1)^{p+1}\alpha_{i_0\cdots i_p}\otimes\beta_{i_{p+2}\cdots i_{p+q+1}})$$

$$f\Big(\sum_{k=0}^{p+1}\alpha_{i_0\cdots\hat{i}_k\cdots i_{p+1}}\otimes\beta_{i_{p+1}\cdots i_{p+q+1}}\Big)$$