$\ulcorner$

$$\|G(\psi)\|_{s+2}^2 = \sum_{\xi\neq 0}(1+\|\xi\|^2)^{s+2}\left|\frac{\psi_\xi}{\|\xi\|^2}\right|^2$$

$$= \sum_{\xi\neq 0}\frac{(1+\|\xi\|^2)^{s+2}}{\|\xi\|^4}|\psi_\xi|^2 = \sum_{\xi\neq 0}\frac{(1+\|\xi\|^2)^2}{\|\xi\|^4}(1+\|\xi\|^2)^s|\psi_\xi|^2$$

$$< \sum_{\xi\neq 0} 2(1+\|\xi\|^2)^s|\psi_\xi|^2 \leq 2\sum_{\xi\in\mathbb{Z}^n}(1+\|\xi\|^2)^s|\psi_\xi|^2 = 2\|\psi\|_s^2 \lrcorner$$

$\Rightarrow$ G is bounded

In case, ψ is perpendicular to the harmonic space, (weak solution space of $\Delta\varphi=0$)

$\varphi = G(\psi)$ gives a weak solution to (*). By the Sobolev lemma, if $\psi\in C^\infty(T)$, then $\varphi\in C^\infty(T)$ and φ is a solution of (*) in the usual sense.

$\ulcorner$ $G(\psi) = \varphi \in H_{s+2}(T)$. $\cdots$ $\varphi\in\bigcap H_s = H_\infty$ $\lrcorner$

Finally, by the Rellich lemma,

$G: L^2(T) \longrightarrow L^2(T)$ is a compact, self-adjoint operator.

$\ulcorner$ $G: L^2(T) = H_0 \xrightarrow{\text{bounded}} H_2(T) \xhookrightarrow{\text{compact}} H_0(T) = L^2(T)$

$$\langle G(\psi), \phi\rangle \overset{?}{=} \langle \psi, G(\phi)\rangle, \psi, \phi\in L^2(T)$$

$$\psi = \sum \psi_\xi e^{i\langle\xi,x\rangle}, \qquad \phi = \sum \phi_\xi e^{i\langle\xi,x\rangle}$$

$$G(\psi) = \sum_{\xi\neq 0}\psi_\xi\frac{e^{i\langle\xi,x\rangle}}{\|\xi\|^2} \qquad G(\phi) = \sum_{\xi\neq 0}\frac{\phi_\xi e^{i\langle\xi,x\rangle}}{\|\xi\|^2}$$

$$\Rightarrow \langle G(\psi),\phi\rangle = \left\langle \sum_{\xi\neq 0}\psi_\xi\frac{e^{i\langle\xi,x\rangle}}{\|\xi\|^2}, \sum\phi_{\xi'}e^{i\langle\xi',x\rangle}\right\rangle$$

$$= \sum_{\xi\neq 0}\psi_\xi\overline{\phi_\xi}\frac{1}{\|\xi\|^2} \overset{?}{=} \langle \psi, G(\phi)\rangle$$

$$= \left\langle \sum_\xi \psi_\xi e^{i\langle\xi,x\rangle}, \sum_{\xi'\neq 0}\frac{\phi_{\xi'}}{\|\xi'\|^2}e^{i\langle\xi',x\rangle}\right\rangle = \sum_{\xi\neq 0}\frac{\psi_\xi\overline{\phi_\xi}}{\|\xi\|^2} \lrcorner$$