Let $\varphi \in A^p(M)$ be a C^∞-p-form and $\sigma = \Sigma a_i f_i$ a piecewise smooth p-chain; we set

$$\langle \varphi, \sigma \rangle = \int_\sigma \varphi = \sum_i a_i \int_\Delta f_i^* \varphi .$$

If φ is a closed form, then, for σ the boundary of a $(p+1)$-chain τ, by Stoke's theorem

$$\int_\sigma \varphi = \int_{\partial\tau} \varphi = \int_\tau d\varphi = 0, \quad \text{so that} \quad \varphi \text{ defines}$$

a real-valued singular p-cocycle.

Again by Stoke's theorem, we have for σ a cycle

$$\int_\sigma \varphi = \int_\sigma \varphi + d\eta = \int_\sigma \varphi + \int_{\partial\sigma} \eta \overset{=0}{} \quad \text{for any } \eta \in A^{p-1}(M);$$

thus there is a map

$$\begin{aligned} H^*_{DR}(M) &\longrightarrow H^*_{sing}(M, \mathbb{R}). \\ \varphi &\longmapsto \int \varphi \end{aligned}$$

The deRham theorem says that this map is in fact an isomorphism.

(proof of de Rham theorem)

First, since any differentiable manifold M can be realized as the underlying topological space of a simplicial complex K, we have

$$H^*_{sing}(M, \mathbb{R}) \cong H^*(K, \mathbb{R}) \cong \check{H}^*(M, \mathbb{R}).$$

Second, by the ordinary Poincare lemma, the sequence of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{A}^0 \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \longrightarrow \mathcal{A}^3 \longrightarrow \cdots$$