$$\Rightarrow \quad \theta = (\partial h)h^{-1}. \qquad \overline{\partial h} = h\,{}^t\overline{\theta} \quad \Rightarrow \quad \partial\overline{h} = \overline{h}\,{}^t\theta$$

$$\Rightarrow \quad {}^t\theta = \overline{h}^{-1}\,\partial\overline{h} \quad \Rightarrow \quad \theta = \partial\,{}^t\overline{h}\;{}^t\overline{h}^{-1}.$$

since $h \underset{\Uparrow}{=} {}^t\overline{h}$, $\quad \theta = \partial h\; h^{-1}$.

$$h_{ij} = (e_i, e_j) = \langle \overline{e_j, e_i} \rangle = \overline{h_{ji}}.$$

$\Rightarrow$ We see that $\theta = \partial h \cdot h^{-1}$ is the unique solution to both equations. Since θ is determined by the conditions of compatibility, θ is well-defined globally. Q.E.D.

$\ulcorner$ $\theta' = \partial h'\, h'^{-1}$. $\quad \theta'$ matrix of one-forms w.r.t $\{e' = e_1' \cdots e_n'\}$

$$\Rightarrow \quad e_i' = \sum g_{ij}(z)\, e_j.$$

$$h'_{ij} = (e_i', e_j') = \langle \sum g_{il} e_l, \sum g_{jk} e_k \rangle = \sum g_{il}\,\overline{g}_{jk}\, h_{lk}$$

$$= (g\, h\, {}^t\overline{g})_{ij} \quad \Rightarrow \quad h' = g\, h\, {}^t\overline{g}.$$

$$\Rightarrow \theta' = \partial h'\, h'^{-1} = \partial(g h\, {}^t\overline{g})\,(g h\, {}^t\overline{g})^{-1} = (\partial g\; h\, {}^t\overline{g} + g\, \partial h\, {}^t\overline{g}$$
$$+ g h \cancel{(\partial\, {}^t\overline{g})}^{=0}\;)(g h\, {}^t\overline{g})^{-1} \quad (\because\ g \text{ is holomorphic} \Leftrightarrow \overline{\partial} g = 0 \Leftrightarrow \partial\overline{g} = 0)$$

$$= (\partial g\, h\, {}^t\overline{g} + g\,\partial h\, {}^t\overline{g})(g h\, {}^t\overline{g})^{-1} = \partial g\; h\, \cancel{{}^t\overline{g}\,({}^t\overline{g})^{-1}}\, h^{-1} g^{-1}$$
$$+ g\,\partial h\, {}^t\overline{g}\,({}^t\overline{g})^{-1}\, h^{-1} g^{-1} = \partial g\; g^{-1} + g\, \partial h\, h^{-1} g^{-1}.$$

since $\partial g = dg = \partial g + \cancel{\overline{\partial} g}^{=0}$, $\quad \theta' = dg\, g^{-1} + g\,\partial h\, h^{-1} g^{-1}$
$$= dg\, g^{-1} + g\,\theta\, g^{-1}. \;\lrcorner$$

The unique connection compatible with the complex and metric structures on E is called the associated, or metric connection. As mentioned in the proof, its matrix w.r.t. to a holomorphic frame is of type (1,0); on the other hand if $e_1 \cdots e_n$ is a unitary frame,

$$0 = d(e_i, e_j) = \theta_{ij} + \overline{\theta_{ji}},$$

so its matrix w.r.t a unitary frame is skew-hermitian.