

Writing out the conditions that e be a cocycle gives the relations:

$$(1) \quad \partial^* \varphi = 0 \Rightarrow \varphi_\alpha \in H^0(U_\alpha, \text{Hom}_0(\mathcal{E}_1/\mathcal{E}_2, \mathcal{F})) \\ \Rightarrow \varphi_\alpha \text{ defines an extension}$$

$$(\mathcal{E}_\alpha) \quad 0 \rightarrow \mathcal{F}|_{U_\alpha} \rightarrow \mathcal{E}_\alpha \rightarrow \mathcal{G}|_{U_\alpha} \rightarrow 0$$

by the same argument as in the proof of the first lemma in the preceding section,

$$(2) \quad -\delta\varphi = \partial^* \eta \Rightarrow \varphi_\alpha - \varphi_\beta = \partial^* \eta_{\alpha\beta} \\ \Rightarrow \text{the local extensions } (\mathcal{E}_\alpha) \text{ given above patch together in double intersections } U_\alpha \cap U_\beta; \text{ and}$$

$$(3) \quad \delta\eta = 0 \Rightarrow \text{the cocycle rule for the patchings of the local extensions in double intersections.}$$

Admittedly, step 3 needs some amplification, but the details to be checked are straightforward enough. Q. E. D.

$$\begin{aligned} \mathbb{F} \quad D e &= D(\varphi \oplus \eta) = D\varphi + D\eta \\ &= (-\partial^* \varphi, \delta\varphi + \partial^* \eta, \delta\eta) \in \end{aligned}$$