$\frac{\partial f}{\partial x_2}(a,b)\, x_2 = 0.$ $\Rightarrow$ $\frac{\partial^2 G}{\partial x_1^2}$, $\frac{\partial^2 G}{\partial x_1 \partial x_2}$ and $\frac{\partial^2 G}{\partial x_2^2}$ can not be zero on some nbd of (a,b), otherwise $\frac{\partial f}{\partial x_1} = C_1$ and $\frac{\partial f}{\partial x_2} = C_2$ $\Rightarrow$ Contredicttion to the fact that $x_3 = f(x_1, x_2)$ is of degree 4.

$\Rightarrow$ The tangent cone is given by

$$\frac{\partial^2 f}{\partial x_1^2}(a,b)\, x_1^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(a,b)\, x_1 x_2 + \frac{\partial^2 f}{\partial x_2^2}(a,b)\, x_2^2 = 0$$

$$\Rightarrow \left\{ \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}(a,b)\right)^2 - \frac{\partial^2 f}{\partial x_1^2}(a,b)\,\frac{\partial^2 f}{\partial x_2^2}(a,b) = 0 \right\}$$

is a subvariety of S s.t $S \cap h$ is a conic curve in h with multiplicity 2.

$\lrcorner$

By the genus formula, then

$$g(B_L) = g(C_L) = 2,$$

so B_L is smooth. (Note that since $\pi(B_L) = 2$ implies a priori that $g(C_L) = g(B_L)$ is less than or equal to 2, this affords another proof that h_L is tangent to S, i.e., that S and S^* are dual.)

$\ulcorner$ $\tilde{C}_L \longrightarrow C_L$ is the desingularization.

$\Rightarrow$ The virtual genus $g(C_L) =$ $g(\tilde{C}_L) = \frac{(4-1)(4-2)}{2} - 1 = 2$ by P280. Let $\tilde{B}_L$ be the desingularization of B_L.

$\Rightarrow$ $\tilde{B}_L \xrightarrow{k} B_L \xrightarrow{j} C_L \subset \mathbb{P}^2$ $\Rightarrow$ $j \circ k : \tilde{B}_L \longrightarrow \mathbb{P}^2$