the first argument that the points $P_2, \cdots P_7$ impose six conditions ; if $P_1 \cdots P_7$ fail to impose seven, every cubic passing through $P_2, \cdots P_7$ will contain $P_1$.

⌜ Choose $P_1'$ s.t $P_1', P_2, \cdots P_7$ have no five collinear points. $\Rightarrow$ By the argument above, $P_1', P_2 \cdots P_7$ impose 7 independent conditions. $\Rightarrow$ $P_2 \cdots P_7$ impose 6 independent conditions. If $P_2$ is a smooth point of $\sigma = a_1\sigma_1 + \cdots + a_{10}\sigma_{10}$, $\frac{\partial \sigma}{\partial X_0}(P_2) X_0 + \frac{\partial \sigma}{\partial X_1}(P_2) X_1 + \frac{\partial \sigma}{\partial X_2}(P_2) X_2 = 0$ is equal to the line with the tangent line $P_1$ passing $P_2$, which gives a condition on $a_1, \cdots a_{10}$. Better way of look.

$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \cdot (1, y') = 0$, where $f(x,y)$ is the homogeneous polynomial of $\sigma$. $\Rightarrow$ $y' = -\frac{\partial f}{\partial x}(P_2) / \frac{\partial f}{\partial y}(P_2) = P_1$

$\Rightarrow$ $\exists$ a linear condition on $a_1, \cdots a_{10}$.

If $P_2$ is a singular point, then

$$\left(\frac{\partial \sigma}{\partial X_0}(P_2), \frac{\partial \sigma}{\partial X_1}(P_2), \frac{\partial \sigma}{\partial X_2}(P_2)\right) = 0 .$$

$\Rightarrow$ $\frac{\partial \sigma}{\partial X_0}(P_2) = 0$ gives a linear condition on $a_1, \cdots a_{10}$.

i.e., $a_1 \frac{\partial \sigma_1}{\partial X_0} + a_2 \frac{\partial \sigma_2}{\partial X_0} + \cdots + a_{10} \frac{\partial \sigma_{10}}{\partial X_0} = 0$ at $P_2$

$\Rightarrow$ $\left(\frac{\partial \sigma_1}{\partial X_0}, \cdots \frac{\partial \sigma_{10}}{\partial X_0}\right) \in \langle (\sigma_1(P_2), \cdots \sigma_{10}(P_2)), \cdots (\sigma_1(P_7), \cdots \sigma_{10}(P_7)) \rangle$

Similarly, we have the same results for $\frac{\partial \sigma}{\partial X_1}(P_2)$ & $\frac{\partial \sigma}{\partial X_2}(P_2)$.

$\Rightarrow$ $\sigma = 0$ is singular at $P_2$ $\Rightarrow$ $\sigma$ contains $P_1$. ⌟