$$\mathcal{E}^*(U) \xrightarrow{\alpha|_U} I(U)$$

$$\begin{aligned}\mathcal{E}(U) &\xrightarrow{\varphi_U} \mathcal{O}(U)\oplus\mathcal{O}(U)\\ \sigma_1 &\longmapsto (1,\ 0) = e_1\\ \sigma_2 &\longmapsto (0,\ 1) = e_2\end{aligned}$$

Let $\sigma_i^*(\sigma_j) = \delta_{ij}$.

Consider $(\alpha|_U(\sigma_1^*),\ \alpha|_U(\sigma_2^*)) \in I(U)\oplus I(U) \subset \mathcal{O}(U)\oplus\mathcal{O}(U)$.

Claim: $\{(\alpha|_U(\sigma_1^*),\ \alpha|_U(\sigma_2^*))\}$ defines a section in $H^0(S, \mathcal{E})$.

$$\begin{array}{ccccc} \mathcal{O}(V\cap U)\oplus\mathcal{O}(V\cap U) & \xleftarrow{\varphi_V} & \mathcal{E}(U\cap V) & \xrightarrow{\varphi_U} & \mathcal{O}(U\cap V)\oplus\mathcal{O}(U\cap V) \\ e_1 & \longleftarrow\!\shortmid\ \sigma_1' & \sigma_1 & \longmapsto & e_1 \\ e_2 & \longleftarrow\!\shortmid\ \sigma_2' & \sigma_2 & \longmapsto & e_2 \\ b_{ij}e_j & \longleftarrow & & & e_i \end{array}$$

Let $\sigma_1' = a_{11}\sigma_1 + a_{12}\sigma_2 \quad \sigma_2' = a_{21}\sigma_1 + a_{22}\sigma_2$

$\sigma_1 = b_{11}\sigma_1' + b_{12}\sigma_2' \quad \sigma_2 = b_{21}\sigma_1 + b_{22}\sigma_2$.

$\Rightarrow\ AB = BA = id. \quad B = (b_{ij}) \quad A = (a_{ij})$

Suppose $\sigma_1^* = c_{11}\sigma_1'^* + c_{12}\sigma_2'^*$ & $\sigma_2^* = c_{21}\sigma_1'^* + c_{22}\sigma_2'^*$.

$\Rightarrow\ \sigma_i^*(\sigma_j) = c_{ij}b_{jk} = \delta_{ij} \quad \Rightarrow\ C = A$.

$\Rightarrow\ \sigma_1^* = b_{11}\sigma_1'^* + b_{12}\sigma_2'^*$

$\sigma_2^* = b_{21}\sigma_1'^* + b_{22}\sigma_2'^*$.

$$\begin{array}{ccc} \mathcal{O}(U\cap V)\oplus\mathcal{O}(U\cap V) & \xrightarrow{\psi} & \mathcal{O}(V\cap U)\oplus\mathcal{O}(V\cap U) \\ (\alpha|_U(\sigma_1^*),\ \alpha|_U(\sigma_2^*)) & \longmapsto & (\alpha|_V(\sigma_1'^*),\ \alpha|_V(\sigma_2'^*)) \end{array}$$

$$= (b_{11}\alpha|_V(\sigma_1'^*) + b_{12}\alpha|_V(\sigma_2'^*),\ b_{21}\alpha|_V(\sigma_1'^*) + b_{22}\alpha|_V(\sigma_2'^*))$$