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$$C_n * d\varphi \wedge \frac{dr}{r^{n-1}} = \pm C_n \frac{dr}{r^{n-1}} \wedge * d\varphi = \pm C_n \langle \frac{dr}{r^{n-1}}, d\varphi \rangle \Phi$$

$$= \pm C_n \langle d\varphi, \frac{dr}{r^{n-1}} \rangle \Phi .$$

$$\Rightarrow \pm C_n * d\varphi \wedge \frac{dr}{r^{n-1}} = C_n \, d\varphi \wedge \frac{*(r dr)}{r^n}$$

$$\pm C_n \int_{B[\delta,\epsilon]} * d\varphi \wedge \frac{dr}{r^{n-1}} = C_n \int_{B[\delta,\epsilon]} d\varphi \wedge \frac{*(r dr)}{r^n} = C_n \int_{B[\delta,\epsilon]} d\eta$$

$$= C_n \int_{\|x\|=\epsilon} \eta - C_n \int_{\|x\|=\delta} \eta = C_n \int_{\|x\|=\epsilon} \varphi \sigma - C_n \int_{\|x\|=\delta} \varphi \sigma$$

We write

$$* d\varphi \wedge \frac{dr}{r^{n-1}} = * d\varphi \wedge d\psi ,$$

where

$$\psi = \begin{cases} \log r & \text{in case } n = 2 \\ \left(-\frac{1}{n-2}\right) \frac{1}{r^{n-2}} & \text{in case } n \geq 3 . \end{cases}$$

Now

$$d * d\varphi = \pm \Delta\varphi \, dx = 0,$$

so that $\quad * d\varphi \wedge d\psi = d(\psi * d\varphi).$

$\Gamma \quad d(\psi * d\varphi) = d\psi \wedge * d\varphi + \psi \quad d(* d\varphi)$

$$d\varphi = \sum \frac{\partial \varphi}{\partial x_i} dx_i .$$

$$* d\varphi = \sum * \left(\frac{\partial \varphi}{\partial x_i} dx_i \right) .$$

$$* \left(\frac{\partial \varphi}{\partial x_i} dx_i \right) = (-1)^{i-1} \frac{\partial \varphi}{\partial x_i} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n .$$

$$\Rightarrow * d\varphi = \sum (-1)^{i-1} \frac{\partial \varphi}{\partial x_i} \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$\Rightarrow d * d\varphi = \sum (-1)^{i-1} \frac{\partial^2 \varphi}{\partial x_i^2} \, dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n \; (-1)^{i-1}$$

$$= \sum \frac{\partial^2 \varphi}{\partial x_i^2} \, dx_1 \wedge \cdots \wedge dx_n = -\Delta\varphi \, dx = 0, \text{ since } \Delta\varphi = 0.$$