$\Rightarrow Qa = kQb \Rightarrow$ Since Q is invertible, $a = kb. \Rightarrow [a] = [b] \in F. \subset \mathbb{P}^n$.

$$\begin{array}{ccc} F & \xrightarrow{\mathcal{G}} & \mathbb{P}^{n*} \\ \rotatebox{90}{$\in$} & & \\ [a] & \longmapsto & \{\sum q_{ij} a_j X_i = 0\} \end{array}$$

Let $Qa = b. \Rightarrow$ Since ${}^t a Q a = 0$, and $Q^{-1} b = a$,

${}^t(Q^{-1}b)\, Q\, Q^{-1} b = {}^t b\, {}^t Q^{-1} b = 0$.

$\Rightarrow \mathcal{G}(F)$ is a smooth quadric. ⌟

Note that in this case no hyperplane in $\mathbb{P}^n$ will be tangent to F more than once; so every tangent hyperplane section $T_p(F) \cap F$ of a smooth quadric in $\mathbb{P}^n$ has rank $n-1$, i.e., is the cone through p over a smooth quadric in $\mathbb{P}^{n-2}$.

⌜ Since $\mathcal{G} : F \longrightarrow \mathbb{P}^{n*}$ is isomorphic, if $\mathcal{G}(p) = \mathcal{G}(q)$, then $p = q$. In otherwords, if $T_p(F) = T_q(F)$, then $p = q$. $\Rightarrow$ No hyperplane in $\mathbb{P}^n$ will be tangent to F more than once, i.e. if $H = T_p(F) = T_q(F)$, then $p = q$. Clearly, $T_p(F) \cap F$ is a quadric in $T_p(F) = \mathbb{P}^{n-1}$.

Let $q \in T_p(F) \cap F$, $q \neq p$.

$\Rightarrow T_q(F) \neq T_p(F)$ by the argument above. $\Rightarrow$ At q, F meets $T_p(F)$ transversely. $\Rightarrow T_p(F) \cap F$ is smooth except p and is of $\dim_{\mathbb{C}}$ $n-2$.

$\Rightarrow T_p(F) \cap F$ is a quadric with the singular set $\{p\}$.