We will show that for $\mu \gg 0$, $L+\mu H$ has a nontrivial global holomorphic section s; then if t is any global holomorphic section of $[H]$ over M, s/t^{μ} will be a global meromorphic section of L as desired.

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$$\left(\frac{s}{t^{\mu}}\right) = (s) - (t^{\mu}) = (s) - \mu(t)$$

$$\begin{tikzcd} L\otimes[H]^{\otimes\mu} \arrow[d, shift right] \\ M \arrow[u, shift right, bend right, "s"'] \end{tikzcd} \qquad \begin{tikzcd} L\otimes[H]^{\otimes\mu}\otimes[-H]^{\otimes\mu} \arrow[r, phantom, "\cong"] \arrow[dr] & L \arrow[d] \\ & M \arrow[ul, bend left, "s/t^{\mu}"] \end{tikzcd}$$

$$\left[\left(s/t^{\mu}\right)\right] = \left[(s) - \mu(t)\right] = [(s)]\ [-H]^{\mu} = L\otimes[H]^{\mu}\otimes[-H]^{\mu}$$
$$= L\otimes[\phi]^{\circ} = L$$

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We proceed by induction on $n = \dim M$: assume that for every submanifold $V \subset \mathbb{P}^N$ of dimension less than n and every line bundle $L \longrightarrow V$,

$$H^{0}(V, \mathcal{O}(L+\mu H)) \neq 0 \quad \text{for } \mu \gg 0.$$

By Bertini's theorem, we can find a hyperplane $\mathbb{P}^{N-1} \subset \mathbb{P}^{N}$ with $V = \mathbb{P}^{N-1}\cap M$ smooth; we consider the exact sheaf sequence

$$0 \to \mathcal{O}_M(L+(\mu-1)H) \xrightarrow{\otimes s} \mathcal{O}_M(L+\mu H) \xrightarrow{r} \mathcal{O}_V(L+\mu H) \to 0.$$