Given an exact sequence

$$0 \longrightarrow \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \longrightarrow 0 \quad \text{of sheaves on } M,$$

we have maps.

$$C^p(\underline{U}, \mathcal{E}) \xrightarrow{\alpha} C^p(\underline{U}, \mathcal{F}), \quad C^p(\underline{U}, \mathcal{F}) \xrightarrow{\beta} C^p(\underline{U}, \mathcal{G})$$

that commutes with δ.

"$(\alpha(\delta\sigma))_{i_0 \cdots i_p} = \alpha\left(\sum_{j=0}^{p} (-1)^j \sigma_{i_0 \cdots \hat{i}_j \cdots i_p} | U_{i_0} \cap \cdots \cap U_{i_p} \right)$

$= \sum_{j=0}^{p} (-1)^j \alpha(\sigma_{i_0 \cdots \hat{i}_j \cdots i_p}) | U_{i_0} \cap \cdots \cap U_{i_p}$ since α commutes with restriction maps.

$= \sum_{j=0}^{p} (-1)^j (\alpha(\sigma))_{i_0 \cdots \hat{i}_j \cdots i_p} | U_{i_0} \cap \cdots \cap U_{i_p}$

$= (\delta(\alpha(\sigma)))_{i_0 \cdots i_p}$. (Note, $(\alpha\sigma)_{\alpha_0 \cdots \alpha_p} = \alpha(\sigma_{\alpha_0 \cdots \alpha_p})$)"

Hence. α, β induce maps

$$H^p(M, \mathcal{E}) \xrightarrow{\alpha^*} H^p(M, \mathcal{F}), \qquad H^p(M, \mathcal{F}) \xrightarrow{\beta^*} H^p(M, \mathcal{G})$$

since α, β commute with ρ_φ.

$(\alpha \circ \rho_\varphi(\sigma))_{\beta_0 \cdots \beta_p} = \alpha\left((\rho_\varphi(\sigma))_{\beta_0 \cdots \beta_p} \right)$

$= \alpha\left(\sigma_{\varphi(\beta_0), \cdots \varphi(\beta_p)} | U'_{\beta_0} \cap \cdots \cap U'_{\beta_p} \right) = \alpha(\sigma_{\varphi(\beta_0) \cdots \varphi(\beta_p)}) | U'_{\beta_0} \cap \cdots \cap U'_{\beta_p}$

$= (\alpha(\sigma))_{\varphi(\beta_0) \cdots \varphi(\beta_p)} | U'_{\beta_0} \cap \cdots \cap U'_{\beta_p}$

$= \rho_\varphi(\alpha(\sigma))_{\beta_0 \cdots \beta_p} \implies \alpha \circ \rho_\varphi = \rho_\varphi \circ \alpha$.

$$\begin{tikzcd}
H^p(\underline{U}, \mathcal{F}) \arrow[r, "\rho^*"] & H^p(\underline{U}', \mathcal{F}) \arrow[r] & \cdots \arrow[r] & H^p(M, \mathcal{F}) \\
H^p(\underline{U}, \mathcal{E}) \arrow[u, "\alpha"] \arrow[r, "\rho"] \arrow[urrr] & H^p(\underline{U}', \mathcal{E}) \arrow[u, "\alpha"] \arrow[r] \arrow[urr] & \cdots \arrow[r] & H^p(M, \mathcal{E}) \arrow[u]
\end{tikzcd}$$