

$$= \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int \int \dots \int_{|z_i|=1} \frac{\sum g_{\bar{i}_1 \dots \bar{i}_n} z_1^{\bar{i}_1} \dots z_n^{\bar{i}_n} z_1^{l_1} \dots z_n^{l_n}}{z_1^{k_1+1} \dots z_n^{k_n+1}} dz_1 \wedge \dots \wedge dz_n$$

$$= \left(\frac{1}{2\pi\sqrt{-1}} \right)^n \int \dots \int \sum g_{\bar{i}_1 \dots \bar{i}_n} \frac{1}{z_1^{k_1-\bar{i}_1-l_1+1} \dots z_n^{k_n-\bar{i}_n-l_n+1}} dz_1 \wedge \dots \wedge dz_n$$

$$= g_{k_1-l_1, k_2-l_2, \dots, k_n-l_n}, \text{ since unless } k_i - \bar{i}_i - l_i + 1 = 1,$$

$$\int \frac{1}{z_i^{k_i - \bar{i}_i - l_i + 1}} dz_i = 0.$$

□

Thus, $\text{res}_f(g, h) = 0$ for all h is equivalent to $g_{\bar{i}_1 \dots \bar{i}_n} = 0$ for $\bar{i}_1 \leq k_1, \dots, \bar{i}_n \leq k_n$, in which case $g \in \{z_1^{k_1+1}, \dots, z_n^{k_n+1}\}$.

This proves the local duality theorem in this case.

□ $\text{res}_f(g, h) = g_{k_1-l_1, \dots, k_n-l_n} = 0$ Since we can choose any $l_1, \dots, l_n \geq 0$, $g_{\bar{i}_1 \dots \bar{i}_n} = 0$ if $\bar{i}_i \leq k_i$.

$$\Rightarrow g(z) = \frac{1}{z_1^{k_1+1} \dots z_n^{k_n+1}} \sum g'_{\bar{i}_1 \dots \bar{i}_n} z_1^{\bar{i}_1} \dots z_n^{\bar{i}_n} \in \{z_1^{k_1+1}, \dots, z_n^{k_n+1}\}$$

$$g'_{\bar{i}_1 \dots \bar{i}_n} = g_{k_1+\bar{i}_1+1, \dots, k_n+\bar{i}_n+1} \quad \square$$

Step Two. We will use the transformation law to prove the

Lemma. Let $f'_1, f'_2, \dots, f'_n \in \mathcal{O}$ and set $f' = (f'_1, f'_2, \dots, f'_n)$, $f = (f_1, \dots, f_n)$. Assume that $f'^{-1}(0) = \{0\} = f'^{-1}\{0\}$ and that $f'_i \in \{f_1, \dots, f_n\}$; i.e., $I(f') \subseteq I(f)$. Then, if the residue pairing is nondegenerate for f' , it is nondegenerate for f .