The exact sheaf sequences

$$0 \to \mathcal{Z}_{\bar\partial}^{p,q}(E) \longrightarrow \mathcal{A}^{p,q}(E) \xrightarrow{\bar\partial} \mathcal{Z}_{\bar\partial}^{p,q+1}(E) \to 0.$$

give us isomorphisms

$$H^i(M, \mathcal{Z}_{\bar\partial}^{p,q+1}(E)) \xrightarrow{\delta} H^{i+1}(M, \mathcal{Z}_{\bar\partial}^{p,q}(E)),$$

since the sheaves $\mathcal{A}^{p,q}(E)$ admit partitions of unity and hence have no Čech cohomology.

$\ulcorner$ $U \subset M$ open.

$\mathcal{Z}_{\bar\partial}^{p,q+1}(E)(U) \ni \sigma$. $\Rightarrow \sigma$ is $\bar\partial$-closed E-valued $(p, q+1)$-form.

$\Rightarrow \sigma = \sum \omega_i \otimes \sigma_i$, σ_i: holomorphic E-valued section (frame)
$\omega_i \in A^{p,q+1}(U)$.

$\Rightarrow \bar\partial\sigma = 0 = \sum \bar\partial\omega_i \otimes \sigma_i = 0 \Rightarrow \bar\partial\omega_i = 0.$

$\Rightarrow \exists\ \tau_i$ s.t $\bar\partial\tau_i = \omega_i$ on V which is, in general, contained in U, by $\bar\partial$-Poincare lemma.

$\Rightarrow \bar\partial(\sum \tau_i \otimes \sigma_i) = \sum \omega_i \otimes \sigma_i$

$\Rightarrow \bar\partial$ is onto. Similarly, we can show that $\ker\bar\partial$ is $\mathcal{Z}_{\bar\partial}^{p,q}(E)$. $\Rightarrow$ The sequence is exact.

$$\Rightarrow \underset{\substack{H^i(M,\, \mathcal{A}^{p,q}(E)) \\ \| \\ 0}}{} \nearrow H^i(M, \mathcal{Z}_{\bar\partial}^{p,q+1}(E)) \longrightarrow H^{i+1}(M, \mathcal{Z}_{\bar\partial}^{p,q}(E)) \longrightarrow \underset{\substack{\| \\ 0.}}{H^{i+1}(M, \mathcal{A}^{p,q}(E))}$$