The content of (**) is sometimes expressed by saying that, given η, the equation

$\Delta\psi = \eta$ has a solution $\Leftrightarrow$ $\mathcal{H}(\eta)=0$, and then

$\psi = G(\eta)$ is the unique solution satisfying $\mathcal{H}(\psi)=0$.

Γ Uniqueness of G.

$$0 = \Delta G. \qquad \bar\partial G = G\circ\bar\partial, \quad \bar\partial^* G = \bar\partial^* G$$

$$\Rightarrow \quad G = 0.$$

pf). Since G is 0 on $\mathcal{H}^{p,q}$, we have only to prove that $G(\bar\partial\phi)=0$ & $G(\bar\partial^*\psi)=0$.

Let $G(\bar\partial\phi)=\psi$. $\Rightarrow$ $\psi = \bar\partial G\phi \in \bar\partial A^{p,q-1}$.

$\Rightarrow$ $\psi \in \bar\partial A^{p,q-1} \cap \mathcal{H}^{p,q}$ since. $\Delta G(\bar\partial\phi) = \Delta\psi = 0$,

$\Rightarrow$ By the previous argument, $\psi = 0$.

Similarly we can show $G(\bar\partial^*\psi)=0$. $\lrcorner$

Γ ($\Rightarrow$) $\Delta\psi = \eta$ has a solution. $\Rightarrow$

$\psi = \mathcal{H}(\psi) + \Delta G(\psi) = \mathcal{H}(\psi) + G(\eta)$.

$\Rightarrow \Delta\psi = \Delta G(\eta) = G(\Delta\eta) = (I-\mathcal{H})(\eta) = \eta - \mathcal{H}(\eta) = \eta$

$\Rightarrow$ $\mathcal{H}(\eta)=0$.

($\Leftarrow$) $\Delta G(\eta) = \eta - \mathcal{H}(\eta)$ $\Rightarrow$ $\Delta G(\eta) = \eta$. $\Rightarrow \exists$ a solution

$\psi = G(\eta)$. $\Delta\psi = \eta$. //.

Uniqueness of Solution.

Suppose $\exists\ \psi$ s.t $\Delta\psi = 0$. and $\mathcal{H}(\psi)=0$.

$\Rightarrow$ $\psi = \mathcal{H}(\psi) + \Delta G\psi = 0$ $\Rightarrow$ $\psi = 0$. $\lrcorner$