$$= \sum_q (-1)^q \operatorname{trace} f^* |_{H_{\bar\partial}^{n,n-q}(M)}$$

$$= \sum_q (-1)^q \operatorname{trace} f^* |_{H_{\bar\partial}^{0,q}(M)}$$

by Kodaira-Serre duality. The number $\sum_q (-1)^q \operatorname{trace} f^* | H_{\bar\partial}^{0,q}(M)$ is called the holomorphic Lefschetz number of the map f, and is denoted $L(f, \mathcal{O})$.

Pf

$$\varphi_\Delta = \sum_{p,q,\mu} (-1)^{p+q} \varphi_{p,q,\mu,\mu}.$$

$$\varphi_{p,q,\mu,\mu} = \pi_1^* \psi_{p,q,\mu} \wedge \pi_2^* \psi^*_{n-p,n-q,\mu} \in \pi_1^* Z_{\bar\partial}^{p,q} \wedge \pi_2^* Z_{\bar\partial}^{n-p,n-q}.$$

$$\Rightarrow \eta_\Delta^\circ = [\varphi_\Delta^\circ], \quad \varphi_\Delta^\circ = \sum_{q,\mu} (-1)^q \varphi_{0,q,\mu,\mu}. \qquad \varphi_{0,q,\mu,\mu} \in \pi_1^* Z_{\bar\partial}^{0,q} \wedge \pi_2^* Z_{\bar\partial}^{n,n-q}$$

$$\Rightarrow \partial \varphi_{0,q,\mu,\mu} = 0 \ \& \ \bar\partial \varphi_{0,q,\mu,\mu} = 0$$

$$\Updownarrow$$

$$d\varphi_{0,q,\mu,\mu} = 0$$

$$\eta_\Delta^\circ(\Gamma_f) = \int_{\Gamma_f} \varphi_\Delta^\circ$$

$$= \sum_q (-1)^q \int_{\Gamma_f} \sum_\mu \pi_1^* \psi_{0,q,\mu} \wedge \pi_2^* \psi^*_{n,n-q,\mu}$$

$$= \sum_q (-1)^q \int_M \sum_\mu \psi_{0,q,\mu} \wedge f^* \psi^*_{n,n-q,\mu}.$$ (see p421)

$$= \sum_q (-1)^q \operatorname{trace} f^* |_{H_{\bar\partial}^{n,n-q}(M)}$$

($\Rightarrow$ By Kodaira-Serre duality theorem

$$H^{n-q}(M, \Omega^n(E)) \cong H^q(M, \Omega^0(E^*)).$$

since in our case $E = M \times \mathbb{C}$,

$$H_{\bar\partial}^{n,n-q}(M) = H^{n-q}(M, \Omega^n) \cong H^q(M, \Omega^0) \cong H_{\bar\partial}^{0,q}(M) \ )$$

$$= \sum_q (-1)^q \operatorname{trace} f^* |_{H_{\bar\partial}^{0,q}(M)}$$

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