Lemma. M is projective $\Leftrightarrow$ it is free

Proof. Assume M is free — we may as well take $M = \mathcal{O}$ — and let $l_i \in L$ be generators and $k_i \in K$ with $\alpha(k_i) = l_i$.

$$\begin{tikzcd} & M \arrow[d, dashed, "\gamma"'] \arrow[dr, "\beta"] & & \\ & K \arrow[r, "\alpha"'] & L \arrow[r] & 0 \end{tikzcd} \qquad L \ni \{l_i\}$$

$M = \mathcal{O}^{(k)} = \underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{k}$

$\exists\ k_i \in K$ s.t. $\alpha(k_i) = l_i$

If $\beta(1) = \sum_i f_i l_i$, then we may set $\gamma(1) = \sum_i f_i k_i$ to fill in the dotted arrow.

If $\beta(\underbrace{(0, \cdots \underset{j}{1} \cdots)}_{e_j}) = \sum_i f_{ji} l_i$, set $\gamma(e_j) = \sum_i f_{ji} k_i$.

$$\Rightarrow \beta(e_j) = \alpha \circ \gamma(e_j) = \alpha\left(\sum_i f_{ji} k_i\right) = \sum_i f_{ji} l_i$$

Conversely, assume M is projective. Taking $M = L$ and K to be free, we have

$$\begin{tikzcd} \mathcal{O}^{(k)} \arrow[r, "\alpha"] & M \arrow[r] \arrow[l, dashed, bend left, "\gamma"] & 0 \end{tikzcd}.$$

$$\begin{tikzcd} L = M \arrow[d, dashed] \arrow[dr] & & \\ \mathcal{O}^{(k)} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

Take $L = M$ and K to be free. so that if M is