$$\Rightarrow \quad \int_Z C_q(\Theta) \geq 0.$$

Perhaps more interesting are the Schwartz-type inequalities. The simplest of these is

$$\int_Z C_1(\Theta)^2 \geq 2 \int_Z C_2(\Theta),$$

where Z is a two-dimensional analytic subvariety of M. For simplicity of notation we prove this when the rank is 2, omitting the exterior multiplication symbol and summing repeated indices. Then

$$C_1(\Theta)^2 = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 (A^1_\mu \bar{A}^1_\mu + A^2_\mu \bar{A}^2_\mu)(A^1_\lambda \bar{A}^1_\lambda + A^2_\lambda \bar{A}^2_\lambda)$$

$$= \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ -(A^1_\mu A^1_\lambda \bar{A}^1_\mu \bar{A}^1_\lambda + A^2_\mu A^2_\lambda \bar{A}^2_\mu \bar{A}^2_\lambda) + 2 A^1_\mu \bar{A}^1_\mu A^2_\lambda \bar{A}^2_\lambda \right\}$$

$$2\, C_2(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ 2 A^1_\mu \bar{A}^1_\mu A^2_\lambda \bar{A}^2_\lambda - 2 A^1_\mu \bar{A}^2_\mu A^2_\lambda \bar{A}^1_\lambda \right\}.$$

Since $C_q(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^q (-1)^{q(q-1)/2} \sum_{\substack{\alpha_1 < \cdots < \alpha_q \\ \mu_1 \cdots \mu_q}} \operatorname{sgn}\pi \, A^{\alpha_1}_{\mu_1} \wedge \cdots \wedge A^{\alpha_q}_{\mu_q} \wedge \bar{A}^{\alpha_{\pi(1)}}_{\mu_1} \wedge \cdots \wedge \bar{A}^{\alpha_{\pi(q)}}_{\mu_q}$,

$$C_2(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 (-1) \left\{ A^1_\mu \wedge A^2_\lambda \wedge \bar{A}^1_\mu \wedge \bar{A}^2_\lambda - A^1_\mu A^2_\lambda \bar{A}^2_\mu \bar{A}^1_\lambda \right\}$$

$$\Rightarrow 2 C_2(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ 2 A^1_\mu \bar{A}^1_\mu A^2_\lambda \bar{A}^2_\lambda - 2 A^1_\mu \bar{A}^2_\mu A^2_\lambda \bar{A}^1_\lambda \right\}$$

Then

$$C_1(\Theta)^2 - 2 C_2(\Theta) = \left(\frac{\sqrt{-1}}{2\pi}\right)^2 \left\{ A^1_\mu \bar{A}^1_\mu A^1_\lambda \bar{A}^1_\lambda + A^2_\mu \bar{A}^2_\mu A^2_\lambda \bar{A}^2_\lambda - 2 A^1_\mu \bar{A}^2_\mu A^2_\lambda \bar{A}^1_\lambda \right\}$$