Hypercohomology

This is a useful generalization of ordinary sheaf cohomology. On a topological space X, a complex of sheaves $(\mathcal{K}^*, d)$ is given by sheaves of Abelian sheaves $\mathcal{K}^p$ together with sheaf maps

$$\mathcal{K}^0 \longrightarrow \cdots \longrightarrow \mathcal{K}^p \xrightarrow{d} \mathcal{K}^{p+1} \longrightarrow \cdots$$

satisfying $d^2 = 0$. In this discussion the notation does not mean that the sheaf sequence is exact. We sometimes write

$$(\mathcal{K}^*, d) = \{ \mathcal{K}^0 \xrightarrow{d} \mathcal{K}^1 \xrightarrow{d} \mathcal{K}^2 \to \cdots\cdots \}.$$

Associated to a complex of sheaves $(\mathcal{K}^*, d)$ are the cohomology sheaves $\mathcal{H}^q = \mathcal{H}^q(\mathcal{K}^*)$: Setting $H^0(U, \mathcal{K}^q) = \mathcal{K}^q(U)$, the presheaf

$$U \longmapsto \frac{\ker\{d : \mathcal{K}^q(U) \to \mathcal{K}^{q+1}(U)\}}{d\, \mathcal{K}^{q-1}(U)}$$

gives rise to a sheaf $\mathcal{H}^q$ whose stalk is

$$\mathcal{H}^q_x = \lim_{U \ni x} \frac{\ker\{d : \mathcal{K}^q(U) \longrightarrow \mathcal{K}^{q+1}(U)\}}{d\, \mathcal{K}^{q-1}(U)} .$$

$\ulcorner$ $U \subset V$, $\qquad H^0(U, \mathcal{K}^q) \xleftarrow{r} H^0(V, \mathcal{K}^q)$ restriction

$$\begin{array}{ccc} H^0(U, \mathcal{K}^q) & \xleftarrow{r} & H^0(V, \mathcal{K}^q) \\ \| & & \| \\ \mathcal{K}^q(U) & & \mathcal{K}^q(V) \end{array}$$

$$\begin{array}{ccc} \mathcal{K}^q(U) & \xrightarrow{d} & \mathcal{K}^{q+1}(U) \\ r\uparrow & & \uparrow r \\ \mathcal{K}^q(V) & \xrightarrow{d} & \mathcal{K}^{q+1}(V) \end{array} \qquad \frac{\ker\{d : \mathcal{K}^q(V) \to \mathcal{K}^{q+1}(V)\}}{d\,\mathcal{K}^{q-1}(V)} \xrightarrow[r]{\text{restriction}} \frac{\ker d}{d\,\mathcal{K}^{q-1}(U)}$$