More precisely,

$$\text{Set of Line Bundles} \xrightarrow{\ \phi\ } H^1(M, \mathcal{O}^*)$$

$$L \longmapsto \overrightarrow{\{g_{\alpha\beta}\}}.$$

Suppose. L and L' have the same image under ϕ.

$\Rightarrow$ $\exists \{U_\alpha\}$ s.t. $\{g_{\alpha\beta}\}$, $\{g'_{\alpha\beta}\}$ are the same in $H^1(\underline{U}, \mathcal{O}^*)$.

$\Rightarrow$ Their difference is a Cech coboundary $\Rightarrow$ $\{g_{\alpha\beta}\}$ $\{g'_{\alpha\beta}\}$ define the same bundle.

Given an elet $l \in H^1(M, \mathcal{O}^*)$, consider. an elet $l' \in H^1(\underline{U}, \mathcal{O}^*)$
$\Rightarrow$ l' defines a line bundle. where $\overrightarrow{l'} = l$
If we have an element $l'' \in H^1(\underline{U}', \mathcal{O}^*)$ s.t $\overrightarrow{l''} = l$.

$\Rightarrow \exists$ $\{U''_\alpha\}$ s.t $l' = l'' \in H^1(\underline{U}'', \mathcal{O}^*)$. $\Rightarrow$ l' & l'' define the same line bundle.

Note:
$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$$

$$(\rho g)_{1,2} = g_{\rho(1)\rho(2)} \big|_{U_1 \cap U_2}.$$

If $\rho(1) = \rho(2)$, $g = 1 \in \mathbb{C}^*$.
Thus. if $U_1, U_2 \subset U_\alpha$.

$$\Rightarrow \underset{\sim}{\underline{U_1 \times \mathbb{C} \amalg U_2 \times \mathbb{C}}} = (U_1 \cup U_2) \times \mathbb{C}.$$

If $U_1 \subset U_\alpha$. $U_2 \subset U_\beta$.

$$\frac{U_1 \times \mathbb{C} \amalg U_2 \times \mathbb{C}}{\sim} = \frac{U_1 \times \mathbb{C} \amalg U_2 \times \mathbb{C}}{\substack{(x, z) \sim (x, g_{\rho(1)\rho(2)}(x) \cdot z) \\ x \in U_1 \cap U_2}}$$

We have only to choose a refinement $\{U'_\alpha\}$ s.t every.