can not, p_1 must lie on L_{47}, L_{57} and L_{67}; thus p_4, p_5 and p_6 all lie on the line L_{17}, and again we have five collinear points.

⌈ For example, if $p_1 \in L_{24}$, then $\overline{p_1 p_2} = L_{24} = L$. $\Rightarrow$ This contradicts to the assumption that L does not contain p_4. $\Rightarrow$ $p_1 \in L_{47}$, L_{57} or L_{67}. $\Rightarrow$ $\overline{p_1 p_7} \ni p_4, p_5, p_6$ ⌟

The lemma follows readily from this first step. Suppose we have eight points $p_1, p_2, \cdots p_8$ in the plane imposing only seven or fewer conditions on cubics, and assume that no five are collinear. By our first step, then, any seven of the eight points p_i do impose independent conditions, and it follows that any cubic passing through any seven of the points contains them all.

⌈ Let $H^0(\mathbb{P}^2, \mathcal{O}(3H)) = \langle \sigma_1, \cdots \sigma_{10} \rangle$.

Consider $V = \{ (\sigma_1(p_1), \cdots \sigma_{10}(p_1)), (\sigma_1(p_2), \cdots \sigma_{10}(p_2)), \cdots (\sigma_1(p_8), \cdots \sigma_{10}(p_8)) \}$. $\Rightarrow$ Suppose $a_1\sigma_1 + \cdots + a_{10}\sigma_{10} = \sigma$, not all $a_i = 0$ contains $p_1, \cdots p_7$. $\Rightarrow$

$$a_1 \sigma_1(p_1) + \cdots + a_{10} \sigma_{10}(p_1) = 0$$
$$a_1 \sigma_1(p_2) + \cdots + a_{10} \sigma_{10}(p_2) = 0$$
$$\vdots$$
$$a_1 \sigma_1(p_7) + \cdots + a_{10} \sigma_{10}(p_7) = 0$$

$\Rightarrow$ Since $\dim V = 7$, $a_1 \sigma_1(p_8) + \cdots + a_{10} \sigma_{10}(p_8) = 0$.

I.e., $(a_1 \cdots a_{10}) \cdot (\sigma_1(p_8), \cdots \sigma_{10}(p_8)) = 0$ since $(\sigma_1(p_8), \cdots \sigma_{10}(p_8))$ is a linear combination of $\{(\sigma_1(p_i), \cdots \sigma_{10}(p_i))\}_{i=1}^{7}$. ⌟