er rotations, it will suffice to prove the mean value property when $y=0$.

⊢ Let $\quad \psi(x) = \varphi(x+y)$.

$\Rightarrow \quad \Delta\psi = \Delta\varphi = 0 \quad$ since Δ is invariant under translations.

$\Rightarrow$ Once we prove the mean-value property when $y=0$,

$$\psi(0) = \int_{\|x\|=\epsilon} \psi(x)\,\sigma_0(x) = \int_{\|x\|=\epsilon} \psi(x)\,\sigma(x)$$

$$= \int_{\|x\|=\epsilon} \varphi(x+y)\,\sigma(x) = \int_{\|t-y\|=\epsilon} \varphi(t)\,\sigma(t-y) \quad (\overset{\text{Put}}{t = x+y}.)$$

$$= \int_{\|t-y\|=\epsilon} \varphi(t)\,\sigma_y(t) = \int_{\|x-y\|=\epsilon} \varphi(x)\,\sigma_y(x) \quad (\overset{\text{Put}}{t = x})$$

$$= \psi(0) = \varphi(y).$$

⌟

We shall apply Stokes' theorem twice to spherical shells $B[\delta,\epsilon] = \{\delta \leq \|x\| \leq \epsilon\}$. The first time we take the $(n-1)$-form

$$\eta = \varphi\sigma.$$

Since $d\sigma = 0$,

$$d\eta = c_n\, d\varphi \wedge \frac{*(r\,dr)}{r^n} = \pm c_n * d\varphi \wedge \frac{dr}{r^{n-1}},$$

and Stokes' theorem gives

$$(*) \qquad \pm c_n \int_{B[\delta,\epsilon]} * d\varphi \wedge \frac{dr}{r^{n-1}} = \int_{\|x\|=\epsilon} \varphi\sigma - \int_{\|x\|=\delta} \varphi\sigma.$$

⊢ $d\eta = d\varphi \wedge \sigma = d\varphi \wedge c_n \dfrac{*(r\,dr)}{r^n} = c_n\, d\varphi \wedge \dfrac{*(r\,dr)}{r^n}$

$= c_n \dfrac{1}{r^n} \langle d\varphi, r\,dr\rangle\, \Phi \quad = c_n \dfrac{1}{r^{n-1}} \langle d\varphi, dr\rangle\, \Phi$