For small t, if x is a fixed point of f_t, obviously, $\frac{d}{dt} f_t(x) = v(f_t(x)) = 0$
$\Rightarrow v(f_t(x)) = v(x) = 0$, $\Rightarrow$ x is a zero point of v.

Since $f(t, x) = x + \int_0^t v(f(s, x))\, ds$,

if $v(p) \neq 0$, for $|t| < \delta$, $\|x - p\| < \epsilon$, δ, ϵ small enough, $\int_0^t v(f(s,x))\, ds \neq 0 \Rightarrow f(t, x) \neq x$.
, in particular, $f(t, p) \neq p$.

For example, $0 \neq v(p) \in R^1$, for $|t| < \delta$, $\|x - p\| < \epsilon$,
$|v(f(s,x)) - v(p)| < \frac{|v(p)|}{2}$, $\Rightarrow$

$$\int_0^t v(f(s,x))\, ds \neq 0 \Rightarrow f(t,x) \neq x. \Rightarrow f(t,p) \neq p.$$

This implies that $f(t, p) = p \Rightarrow v(p) = 0$. --- ①

Conversely, if $v(p) = 0$, consider the following differential equation

$$(*) \cdots \quad \frac{d}{dt} f(t, p) = v(f(t, p)).$$
$$f(0, p) = p.$$

Consider the diff equt $\frac{d}{dt} f(t, x) = v(f(t, x))$. $\Rightarrow \exists$ a unique solution $f(t, x)$ for $|t| < \delta$, $\|p - x\| < \epsilon$, $f(t, x)|_{x=p}$ is the solution for $(*)$ by uniqueness

$\Rightarrow$ Obviously, $f(t, p) = p$ is a solution for the equation, since $LHS = \frac{dp}{dt} = 0$ $RHS = v(f(t,p)) = v(p) = 0$.
(This gives an information on the behavior of f near p for $|t| < \delta$.)
$\Rightarrow$ By the uniqueness, $f(t, p) = p$ is the solution, which implies $f_t(p) = p$ for $|t| < \delta$, δ small enough. .. ②
(See Spivak Vol I Chap. 5).

By ① & ②, for each point p s.t $v(p) = 0$ or $f_t(p) = p$ for $|t| < \delta$, $v(p) = 0 \Leftrightarrow f_t(p) = p$.

Since such p is isolated, and M is compact, we have a finite # of such p's and we get $\epsilon > 0$ s.t for $|t| < \epsilon$, the fixed points of f_t are exactly the zeros of v.

By using Taylor series, & $f(0, x) = x$,