$\Rightarrow$ This implies that $\{ i_K(P_1) = [\omega_1(P_1), \cdots \omega_g(P_1)] \cdots i_K(P_\ell) = [\omega_1(P_\ell), \cdots \omega_g(P_\ell)] \} \subset \mathbb{P}^{g-1}$ a set of points which are linear independent. $\Rightarrow \dim \langle \omega_1 \cdots \omega_g \rangle \geq \ell$.

The argument above has nothing to do with the proof.

Consider the following observations.
Given a hyperplane $H$ containing $E$, then we have an effective divisor in $|D|$ whose a linear subseries is cut out by $H$.
Suppose $H_1$ & $H_2$ are linearly independent hyperplanes containing $E$. Assume that $D_1$ & $D_2$ are effective divisors in $|D|$ whose linear subseries are cut out by $H_1$ & $H_2$ respectively. Then we claim that $D_1 \neq D_2$ as divisors. If $D_1 = D_2$. $\Rightarrow E + D_1 = E + D_2$
$\Rightarrow H_1 \cap i_K(S) = H_2 \cap i_K(S) \Rightarrow$ Since the points of $H_i \cap i_K(S)$ span $H_i$ for each $i$, $H_1 = H_2 \Rightarrow *$.
If $H_1 \cap i_K(S)$ do not span $H_1$, we have a point $p \notin H_1 \cap i_K(S)$ s.t $\overline{p, H_1 \cap i_K(S)} \cap i_K(S) > \deg (i_K(S)) = \#(H_1 \cdot i_K(S))$.

Thus we have a unique effective divisor in $|D|$ corresponding to a hyperplane $H \supset E$.
$\Rightarrow$ Since $\dim \overline{E} \leq g - s - 2$, there are at least $\dim G(g-(g-s-1), s+2)$ number of hyperplanes which have linearly independent.
$\Rightarrow$ Since $\dim G(s+1, s+2) = s$, we have at least $s$ number of effective divisors in $|D|$, which implies $\dim H^0(S, \mathcal{O}(D)) = h^0(D) \geq s$. ~~linear indep~~

$\dim G(s+1, s+2) = \#$ of choices of choosing $(s+2)$-plane in $\mathbb{C}^{s+2}$.