Since $H^1(\mathcal{F}(k))$ is finite dimensional, we must have isomorphisms

$$H^1(\mathcal{F}(k)) \longrightarrow H^1(\mathcal{F}(k+1)) \longrightarrow H^1(\mathcal{F}(k+2))$$

for $k \geq k_0$. But then the cohomology sequence of $(**)$ gives

$$H^0(\mathcal{F}(k)) \longrightarrow H^0(\mathcal{F}_H(k)) \longrightarrow 0$$

for $k \geq k_1$.

$\Vdash$

$$\begin{tikzcd} H^1(\mathcal{F}(k-1)) \arrow[r, "\cong"] \arrow[rr, bend right, "\text{isomorph}", "\text{by the above}"'] & H^1(\mathrm{im} f) \arrow[r, two heads] & H^1(\mathcal{F}(k)) \arrow[r] & 0. \end{tikzcd}$$

$\Rightarrow \quad H^1(\mathrm{im} f) \xrightarrow{\cong} H^1(\mathcal{F}(k)).$

$\Rightarrow$ From the following exact sequence

$$H^0(\mathrm{im} f) \longrightarrow H^0(\mathcal{F}(k)) \longrightarrow H^0(\mathcal{F}_H(k)) \longrightarrow H^1(\mathrm{im} f) \xrightarrow{\cong} H^1(\mathcal{F}(k)),$$

$\Rightarrow \quad H^0(\mathcal{F}(k)) \longrightarrow H^0(\mathcal{F}_H(k)) \longrightarrow 0.$ $\lrcorner$

Now $H^0(\mathcal{F}_H(k))$ generates $\mathcal{F}_H(k)_x$ as an $\mathcal{O}_{H,x}$-module for $k \geq k_2$ and any $x \in H$.

$\Vdash$ By induction hypothesis, $\dim_{\mathbb{C}} H = n-1$. $\lrcorner$

Since the tangent space to H at x was assigned arbitrary, it follows easily that $H^0(\mathcal{F}(k))$ generates $\mathcal{F}(k)_x$ as an $\mathcal{O}_{M,x}$-module. Q.E.D.

$\Vdash$ Assume $0 \longrightarrow \mathcal{F} \to 0$ locally for simplicity.