The inverse image $E = \pi^{-1}(p)$ is naturally isomorphic to $\mathbb{P}(T_p(M)) \cong \mathbb{P}^{n-1}$ and is called the exceptional divisor of the blow-up $\tilde{M} \to M$.

See P183 ~ P184

When blow-ups were introduced in the course of the Kodaira embedding theorem, we were primarily concerned with the local geometry of M and $\tilde{M}$ near p, and E respectively. We would now like to relate the global geometry of $\tilde{M}$ to that of M. We begin by considering the topology of M and $\tilde{M}$: we set $M^* = M - \{p\}$, $\tilde{M}^* = \pi^{-1} M^* = \tilde{M} - E$, $U^* = U - \{p\}$ and $\tilde{U}^* = \pi^{-1} U^* = \tilde{U} - E$, and compare the Mayer-Vietoris sequences of $M = M^* \cup U$ and $\tilde{M} = \tilde{M}^* \cup \tilde{U}$:

$$\begin{tikzcd}
H_i(\tilde{U}^*) \arrow[r] \arrow[d, "\pi_*", "\cong"'] & H_i(\tilde{U}) \oplus H_i(\tilde{M}^*) \arrow[r] \arrow[d, "\pi_*", "\cong"'] & H_i(\tilde{M}) \arrow[r] \arrow[d, "\pi_*"] & H_{i+1}(\tilde{U}^*) \arrow[d, "\pi_* \cong"] \\
H_i(U^*) \arrow[r] & H_i(U) \oplus H_i(M^*) \arrow[r] & H_i(M) \arrow[r] & H_{i+1}(U^*).
\end{tikzcd}$$

Now, π_* is an isomorphism between $H_*(\tilde{U}^*)$ and $H_*(U^*)$, and between $H_*(\tilde{M}^*)$ and $H_*(M^*)$.

Since $\pi: \tilde{U}^* \longrightarrow U^*$ and $\pi: \tilde{M}^* \longrightarrow M^*$ are isomorphic, the induced maps $\pi_*: H_*(\tilde{U}^*) \longrightarrow H_*(U^*)$ & $\pi_*: H_*(\tilde{M}^*) \longrightarrow H_*(M^*)$ are isomorphisms.