$$\begin{tikzcd} {[\mathcal{S}\cap H]} \arrow[d] & {[\mathcal{S}_0\cap H]} \arrow[d] \\ \mathcal{S} \arrow[r, "\varphi"] & \mathcal{S}_0 \arrow[u, bend right, "\sigma_0"'] \end{tikzcd}$$

σ_0 a section s.t $(\sigma_0=0)=\mathcal{S}_0\cap H$.

$\Rightarrow (\varphi^*\sigma_0=0)=\mathcal{S}\cap H$ since $\varphi:\mathcal{S}\to\mathcal{S}_0$ carries $\mathcal{S}\cap H$ to $\mathcal{S}_0\cap H$. $\Rightarrow$ $\varphi^*[\mathcal{S}_0\cap H]=\varphi^*(H|_{\mathcal{S}_0})=[\mathcal{S}\cap H]$ $=H|_{\mathcal{S}}$. $\Rightarrow$ $\underset{\varphi^*(K_{\mathcal{S}_0})}{\varphi^*(-H|_{\mathcal{S}_0})}=-H|_{\mathcal{S}}=K_{\mathcal{S}}$.

$\Rightarrow$ $c_1(\varphi^*K_{\mathcal{S}_0})=c_1(K_{\mathcal{S}})=\varphi^*(c_1(K_{\mathcal{S}_0}))$ by P407 ┘

Let $\eta_{E_i}\in H^2(\mathcal{S}_0,\mathbb{Z})$ be the cohomology class of the exceptional divisor $E_i\subset\mathcal{S}_0=\widetilde{\mathbb{P}}^2_{p_1,\dots,p_6}$, and set $\mu_i=\varphi^*\eta_{E_i}$. Since $K_{\mathcal{S}}$ is negative, it clearly can not have any global sections, so $h^{2,0}(\mathcal{S})=h^{0,2}(\mathcal{S})=0$ and the classes $\mu_i\in H^2(\mathcal{S},\mathbb{Z})$ are necessarily of type (1.1).

┌ Since $K_{\mathcal{S}}=-H|_{\mathcal{S}}$, $K_{\mathcal{S}}$ is negative. $\Rightarrow$ By P155, $H^0(\mathcal{S},\mathcal{O}(K_{\mathcal{S}}))=0$. $\Rightarrow$ $H^0(\mathcal{S},\mathcal{O}(K_{\mathcal{S}}))=H^0(\mathcal{S},\Omega^2)=H^{2,0}_{\bar\partial}(\mathcal{S})=\overline{H^{0,2}_{\bar\partial}(\mathcal{S})}=0$ $\Rightarrow$ $h^{2,0}(\mathcal{S})=\dim H^{2,0}_{\bar\partial}(\mathcal{S})=0=\dim H^{2,0}(\mathcal{S})$. By P64, E_i is nonzero in the homology of $\mathcal{S}_0$, and in the cohomology of $\mathcal{S}_0$. $\Rightarrow$ $\eta_{E_i}\neq 0$ $\Rightarrow$ $\eta_{E_i}\in H^2(\mathcal{S}_0)=H^{2,0}(\mathcal{S}_0)\oplus H^{1,1}(\mathcal{S}_0)\oplus H^{0,2}(\mathcal{S}_0)$. $\Rightarrow$ $\eta_{E_i}\in H^{1,1}(\mathcal{S}_0)$. $\Rightarrow$ $\varphi^*\eta_{E_i}\in H^{1,1}(\mathcal{S})$. ┘

By the Lefschetz (1,1) theorem, there exists a holomorphic line bundle $L_i\longrightarrow\mathcal{S}$ with $c_1(L_i)=\mu_i$.