Since $\hat{e}_J(e_{J'}) = \begin{cases} 0 & J \neq J' \\ 1 & J = J' \end{cases}$,

$I \supset J$. If $I \not\supset J$, $a_I = 0$.

$\Rightarrow$ $I = \{ j_1 < \cdots < j_{\nu-1} < j^\circ_\ell < j_\nu < \cdots < j_k \}$. $\Rightarrow$

$a_I = (-1)^{\nu-1} f_{j^\circ_\ell}$

$$\Rightarrow \partial^* \hat{e}_J = \sum_{I=\{j_1<\cdots<j_{\nu-1}<j^\circ_\ell<j_\nu<\cdots j_k\}} (-1)^{\nu-1} f_{j^\circ_\ell} \hat{e}_I$$

$$\Rightarrow \partial^* \hat{e}_J \longmapsto \sum_I (-1)^{\nu-1} f_{j^\circ_\ell} e_{I^*}$$

$$= \sum_I (-1)^{\nu-1} f_{j^\circ_\ell} \epsilon(I, I^\circ) e_{I^\circ}$$

$$= \sum_I (-1)^{\nu-1} f_{j^\circ_\ell} \epsilon(I, I^\circ) e_{\{j^\circ_1 < \cdots < \hat{j}^\circ_\ell < \cdots j^\circ_{r-k}\}}$$

$$= \sum_{\{j^\circ_1<\cdots \hat{j}^\circ_\ell \cdots < j^\circ_{r-k}\} = J^\circ - \{j^\circ_\ell\}} (-1)^{\nu-1} f_{j^\circ_\ell} \epsilon(I, I^\circ)\, e_{J^\circ - \{j^\circ_\ell\}}$$

Since $\epsilon(I, I^\circ) = \epsilon(j_1, \cdots j_{\nu-1}, j^\circ_\ell, j_\nu \cdots j^\circ_1, \cdots \hat{j}^\circ_\ell \cdots j^\circ_{r-k})$

$= (-1)^{\ell-1} (-1)^{k-(\nu-1)} \epsilon(j_1 \cdots j_k, j^\circ_1 \cdots j^\circ_\ell \cdots j^\circ_{r-k}) = (-1)^{\ell+\nu+k} \epsilon(J, J^\circ)$,

$$= \sum_{J^\circ - \{j^\circ_\ell\}} (-1)^{\nu-1} f_{j^\circ_\ell} (-1)^{\ell+\nu+k} \epsilon(J, J^\circ) e_{J^\circ - \{j^\circ_\ell\}}$$

$$= \sum (-1)^{\ell-1} (-1)^k f_{j^\circ_\ell} \epsilon(J, J^\circ) e_{J^\circ - \{j^\circ_\ell\}}$$

$$= (-1)^k \left(\sum (-1)^{\ell-1} f_{j^\circ_\ell} e_{J^\circ - \{j^\circ_\ell\}} \right) \epsilon(J, J^\circ)$$

$$= (-1)^k (\partial e_J)(\epsilon(J, J^\circ)) = (-1)^k \partial(\epsilon(J, J^\circ) e_J)$$

$$= (-1)^k \partial e_{J^*}.$$

$$\Rightarrow \begin{tikzcd} e_J \in \mathrm{Hom}_{\mathcal{O}}(E_k, \mathcal{O}) \arrow[r, "\sim"] \arrow[d, "\partial^*"] & E_{r-k} \ni e_{J^*} \arrow[d, "\partial"] \\ \partial^* e_J \in \mathrm{Hom}_{\mathcal{O}}(E_{k+1}, \mathcal{O}) \arrow[r] & E_{r-k-1} \ni (-1)^k \partial e_{J^*} \end{tikzcd}$$

up to sign $(-1)^k$.