The two operators $\bar\partial$ and $*$ have nothing to with E, i.e. they act only on $f\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q$.

$$\Rightarrow \quad \Delta\psi = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\psi$$

$$\equiv \Big(-2\sum_K f_{\bar K, K}\Big)\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi \otimes e$$ modulo lower order terms.

Lower order terms involve f's of 0 and 1-st order

$$\nabla(f\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q\otimes e)$$

$$= df\otimes(\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q\otimes e) + f\sum a_{Ji}\,\bar\varphi_J\otimes e_i$$

$$\Rightarrow \nabla_{\bar K}(f\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q\otimes e)$$
$$\equiv f_{\bar K}\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q\otimes e \quad (\text{mod lower order terms})$$

$$\Rightarrow \nabla_K\nabla_{\bar K}(f\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q\otimes e) \equiv f_{\bar K, K}\,\bar\varphi_1\wedge\cdots\wedge\bar\varphi_q\otimes e \ (\text{mod. l.o.t})$$

This proves the Weitzenböck formula

$$(\Delta\psi)_{I\bar J, i} = -2\sum_{K=1}^{n}\nabla_K\nabla_{\bar K}\,\psi_{I\bar J i} + A'(\psi)_{I,\bar J, i}$$

where

$$\psi = \frac{1}{p!\,q!}\sum_{I,J,i}\psi_{I,\bar J, i}\,\varphi_I\wedge\bar\varphi_J\otimes e_i\ .$$

We now come to the proof of the Garding inequality, where we assume the Weitzenböck in the form

$$(\Delta\psi)_{I\bar J, i} = -2\sum_K \psi_{I\bar J. i.\bar K. K} + A'(\psi)_{I,\bar J, i}\,.$$