$$\varphi_U : E_U \longrightarrow U \times \mathbb{C}^k \quad (\text{where } E_u = \pi^{-1}(U))$$

induce maps $\varphi_U^* : E_U^* \longrightarrow U \times \mathbb{C}^{k*} \cong U \times \mathbb{C}^k$ which give $E^* = \bigcup E_x^*$ the structure of a manifold.

The construction is most easily expressed in terms of transition functions: if $E \to M$ has transition function $\{g_{\alpha\beta}\}$, then $E^* \to M$ is the complex vector bundle given by transition functions

$$\tilde{j}_{\alpha\beta}(x) = {}^t g_{\alpha\beta}(x)^{-1}$$

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}^k \qquad \varphi_\beta : \pi^{-1}(U_\beta) \longrightarrow U_\beta \times \mathbb{C}^k$$
$$\varphi_\alpha^* : \pi^{-1}(U_\alpha)^* \longrightarrow U_\alpha \times \mathbb{C}^{k*} \qquad \varphi_\beta^* : \pi^{-1}(U_\beta)^* \longrightarrow U_\beta \times (\mathbb{C}^k)^*$$

$$\tilde{J}_{\alpha\beta} = \varphi_\alpha^* \circ \varphi_\beta^{*-1} \qquad g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}.$$

pull-back of v

$\Rightarrow \tilde{j}_{\alpha\beta}(L)(v) = L(g_{\alpha\beta}^{-1} v)$. since L can be represented by a vector, $\quad (\varphi_\beta^{*-1}(L))(\varphi_\alpha(v)) = L(\varphi_\beta^{-1} \circ \varphi_\alpha(v))$

$${}^t(\tilde{j}_{\alpha\beta} L)\, v = {}^t L (g_{\alpha\beta}^{-1} v) = {}^t L\, g_{\alpha\beta}^{-1}\, v$$

$$\Rightarrow \quad {}^t(\tilde{j}_{\alpha\beta} L) = {}^t L\, g_{\alpha\beta}^{-1} \quad \Rightarrow \quad \tilde{j}_{\alpha\beta} L = {}^t g_{\alpha\beta}^{-1} L$$

$$\Rightarrow \tilde{j}_{\alpha\beta} = {}^t g_{\alpha\beta}^{-1}.$$

Similarly, if $E \to M$, $F \to M$ complex vector bundles of rank k and l with transition functions $\{g_{\alpha\beta}\}$ & $\{h_{\alpha\beta}\}$ respectively, then we can define bundles

1, $E \oplus F$ given by transition functions.

$$\tilde{j}_{\alpha\beta}(x) = \begin{pmatrix} g_{\alpha\beta}(x) & 0 \\ 0 & h_{\alpha\beta}(x) \end{pmatrix} \in GL(\mathbb{C}^k \oplus \mathbb{C}^l)$$