and the curves $\{\pi^* D_\lambda\}$ do form a linear system, $\{\tilde{D}_\lambda\}$ will be a linear system if and only if all the curves D_λ have the same multiplicity at p. Thus when we speak of the proper transform of a linear system $\{D_\lambda\}$, we will mean the linear system of curves $\{\pi^* D_\lambda - mE\}$, where $m = \min\{\mathrm{mult}_p(D_\lambda)\}_\lambda$ is the multiplicity of the generic curve D_λ at p.

Γ ($\Leftarrow$) Let $\mathrm{mult}_p(D_\lambda) = m$, $\pi^* D_\lambda = (S_\lambda^* = 0)$ and $\pi^* D_{\lambda'} = (S_{\lambda'}^* = 0)$.

$\Rightarrow \tilde{D}_\lambda = (S_\lambda^* \otimes e^{-m} = 0)$, where $(e=0) = E$.

$\tilde{D}_{\lambda'} = (S_{\lambda'}^* \otimes e^{-m} = 0)$.

$\Rightarrow S_\lambda^* \otimes e^{-m} + S_{\lambda'}^* \otimes e^{-m} = (S_\lambda^* + S_{\lambda'}^*) \otimes e^{-m} = S_{\lambda''}^* \otimes e^{-m}$

, since $\{\pi^* D_\lambda\}$ is a linear system, and $S_\lambda^* + S_{\lambda'}^* = S_{\lambda''}^*$ where $\pi^* D_{\lambda''} = (S_{\lambda''}^* = 0)$. $\Rightarrow (S_{\lambda''}^* \otimes e^{-m} = 0) = (S_{\lambda''}^* = 0) - m(e=0) = \pi^* D_{\lambda''} - mE = \tilde{D}_{\lambda''}$, $\mathrm{mult}_p(D_{\lambda''}) = m$ and $D_{\lambda''} \in \{D_\lambda\}$. $\Rightarrow$ This concludes that $\{\tilde{D}_\lambda\}$ is a linear system.

($\Rightarrow$) Let $D_1 = (f_1 = 0)$, $D_2 = (f_2 = 0)$ locally.

$\Rightarrow f_1 = \sum_{m \geq m_1} (f_1)_m$, $f_2 = \sum_{m \geq m_2} (f_2)_m$, $m_1 = \mathrm{mult}_p(D_1)$, $m_2 = \mathrm{mult}_p(D_2)$.

$\Rightarrow f_1 + f_2 = f_3$

Suppose $m_1 \neq m_2$. $\Rightarrow$ Assume $m_1 < m_2$. $\Rightarrow f_3$ vanishes to order m_1. $\Rightarrow$

$(\pi^* f_1 = 0) = m_1 E + \tilde{D}_1$

$(\pi^* f_2 = 0) = m_2 E + \tilde{D}_2$

$(\pi^* f_3 = 0) = m_1 E + \tilde{D}_3$, $D_3 = (f_3 = 0)$

$\Rightarrow$ By the assumption that $\{\tilde{D}_\lambda\}$ is a linear system,

$\frac{\pi^* f_1}{z_i^{m_1}} + \frac{\pi^* f_{i+1}}{z_i^{m_1}} = \frac{\pi^* f_{i+2}}{z_i^{m_1}} \cdots ⊛$, since $\tilde{D}_1 = \left(\frac{\pi^* f_1}{z_i^{m_1}} = 0\right)$