The converse is harder to prove in general; since we will use it only for analytic hypersurfaces, we will prove it in this case. Suppose V^* is disconnected, and let $\{V_i\}$ denote the connected components of V^*; we want to show that $\overline{V_i}$ is an analytic variety. Let $p \in \overline{V_i}$ be any point, f a defining function for V near p, and $z = (z_1, \cdots z_n)$ local coordinates around p; we may assume that f is a Weierstrass polynomial of degree k in z_n.

Write

$$g = \alpha f + \beta \frac{\partial f}{\partial z_n}, \qquad g \neq 0 \in \mathcal{O}_{n-1};$$

then for Δ some polydisc around p and Δ' a polydisc in $\mathbb{C}^{n-1}$, the projection map $\pi : (z_1, \cdots z_n) \mapsto (z_1, \cdots z_{n-1})$ expresses $V_i \cap (\Delta - (g=0))$ as a covering space of $\Delta' - (g=0)$.

Since f is a local defing function at p, f & $\frac{\partial f}{\partial z_n}$ are relatively prime, otherwise, $f(z', z_n) = (\quad)(\quad)^2(\cdots)$, which is not a local defining function. More precisely, suppose $f = g f_1$ & $\frac{\partial f}{\partial z_n} = g f_2$, where

g is irreducible in $\mathcal{O}_{n-1}[z_n]$.

$$\Rightarrow \frac{\partial f}{\partial z_n} = \frac{\partial g}{\partial z_n} f_1 + g \frac{\partial f_1}{\partial z_n} = g f_2$$

$$\Rightarrow \frac{\partial g}{\partial z_n} f_1 = g\left(f_2 - \frac{\partial f_1}{\partial z_n}\right), \Rightarrow \text{Since } g \text{ is}$$