Suppose now that $E \longrightarrow M$ is a vector bundle over a compact manifold M. Assume that we have connection ∇ in E and in the tangent bundle TM of M. (It is more convenient to denote the connection operator by ∇, rather than D.).

If $\{e_\alpha\}$ is a local frame for E and $\{v_i\}$ a local frame for TM with dual coframe $\{\varphi_i\}$, then the covariant derivatives $\nabla_i f_\alpha = f_{\alpha,i}$ of a section

$$\left(\nabla_{v_i} f\right)_\alpha \ \overset{\shortparallel}{=}\ v_i(f_\alpha)$$

$\nwarrow$ α-th term i.e coefficient of e_α

$f = \sum f_\alpha e_\alpha$ of $E \longrightarrow M$ are defined by

$$\nabla f = \sum f_{\alpha,i}\, e_\alpha \otimes \varphi_i .$$

We have $f_{\alpha,i} = v_i f_\alpha + A^\circ(f)$, where A° is an operator of order zero involving the connection matrix.

Applying these considerations to $E \otimes T^*M$, we may define $f_{\alpha,i,j} = \nabla_j(\nabla_i f_\alpha)$, and so forth. The commutation rule

$[\nabla_i, \nabla_j] f_\alpha = A'(f)$ follows from the above expression for $f_{\alpha,i}$.

$\lceil\Gamma$ $\quad f = \sum f_\alpha e_\alpha \qquad f_{\alpha,i} = (\nabla_{v_i} f)_\alpha$

$$\left(\nabla_{v_j}(\nabla_{v_i} f)\right)_\alpha = (f_{\alpha,i})_j = f_{\alpha,i,j}$$

$$\nabla_{v_i} f = \sum_\alpha f_{\alpha,i}\, e_\alpha \qquad \nabla_{v_j} f = \sum f_{\alpha,j}\, e_\alpha$$

$$\left(\nabla_{v_i}(\nabla_{v_j} f)\right)_\alpha = (f_{\alpha,j})_i = f_{\alpha,j,i} .$$