At $(0,0)$, $dz_2 = \frac{g_2(0,0)}{h(0,0)} dw_2 \Rightarrow dz_2 \neq 0$ at $T_p'M/T_p'C$.

$\Rightarrow$ By choosing $\tilde{U}_1$ small enough so that $d(z_1/z_2)$ & $dz_2$ generate $T_p^{*\prime}M$, $p \in \tilde{U}_1$, since $\{d(z_1/z_2), dz_2\}$ is a base for $T_p^{*\prime}M$ for $p \in U_1$, we may take $z_2$, $z_1/z_2$ local coordinates on $\tilde{U}_1$. Note here that $\{d(z_1/z_2), dz_2\}$ is a base for $T^{*\prime}M \Leftrightarrow \det\begin{bmatrix} \frac{\partial(z_1/z_2)}{\partial w_1}, & \frac{\partial(z_1/z_2)}{\partial w_2} \\ \frac{\partial z_2}{\partial w_1}, & \frac{\partial z_2}{\partial w_1} \end{bmatrix}$

is nonzero. ⏌

Similarly, $z_1$ and $z_2/z_1$ furnish local coordinates on an open set $\tilde{U}_2 \subset M$ containing $U_2$. We see from this that the functions $(z_1, z_2)$ map a neighborhood of $C$ in $M$ onto a neighborhood of the origin in $\mathbb{C}^2$, a mapping that is holomorphic outside. This proves that $\iota_{L'}(C)$ is a smooth point of $\iota_{L'}(M)$, completing the proof of the Castelnuovo-Enriques criterion. Q.E.D.

A smooth rational curve of self-intersection $-1$ on a surface is called an exceptional divisor of the first kind.

⌈

$$\begin{array}{ccccc} \tilde{U}_1 & \xrightarrow{\phi} & \mathbb{C}^2 & \xrightarrow{\psi} & \mathbb{C}^2 \\ \rotatebox{90}{$\in$} & & & & \\ p & \longmapsto & (\frac{z_1}{z_2}(p), z_2(p)) & \longmapsto & (z_1(p), z_2(p)) \end{array}$$

$\Rightarrow \phi(\tilde{U}_1)$ open in $\mathbb{C}^2$, by the argument above.

To show that $\psi$ sends open nbd of $(0,0)$ to an open nbd