If I don't like the way of expressing the one parameter group $f(t, x)$. They said $f(t, x) = \exp(t\, v(x))$.
If $\exp(t\, v(x))$ is the geodesic s.t

$$\exp(0\, v(x)) = x$$

$$\left.\frac{\partial \exp(t\, v(x))}{\partial t}\right|_{t=0} = v(x),$$

for all t.

then $f(t, p) = \exp(t\, v(p)) = p$ in case $v(p) = 0$.
In general, this is not true.

We are going to show that $f(t, z)$ is holomorphic.
For simplicity, let $n = 1$.

$$\frac{\partial f(t,z)}{\partial t} = v(f(t,z))$$

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial t} f(t,z)\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \bar{z}} f(t,z)\right) = \frac{\partial}{\partial \bar{z}} v(f(t,z))$$

$$= \frac{\partial v}{\partial \bar{w}} \frac{\partial f(t,z)}{\partial \bar{z}} = 0 \quad \text{since} \quad \frac{\partial v}{\partial \bar{w}} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \bar{z}} f(t,z)\right) = 0 = \frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial t}(f(t,z))\right) = 0$$

$$\Rightarrow \frac{\partial}{\partial t} f(t,z) \text{ is holomorphic.}$$

Let $\frac{\partial}{\partial t} f(t,z) = a(t,z) + \sqrt{-1}\, b(t,z)$.

$$\Rightarrow \frac{\partial a(t,z)}{\partial x} = \frac{\partial b(t,z)}{\partial y} \Rightarrow \frac{\partial}{\partial x}\int a(t,z)\,dt = \frac{\partial}{\partial y}\int b(t,z)\,dt.$$