To illustrate how nilpotents arise geometrically, suppose that $Z \subset M$ is an irreducible subvariety defined by a sheaf of prime ideals $I \subset \mathcal{O}$. Then the ideal sheaves $I^{\mu+1}$ define spaces $Z_\mu = (Z, \mathcal{O}/I^{\mu+1})$, which may be thought of as the μth infinitesimal nbd of Z.

┌ I don't know whether a subvariety defined by a sheaf of prime ideals is irreducible or not. At least, for a projective space $\mathbb{P}^n$, it is necessary and sufficient, see P4 & P11, Ex. 2.4 (b) Algebraic Geometry by Hartshorne. ┘

In this context let us reexamine the Reiss relation discussed in Section 2. Here $M = \mathbb{P}^2$ and $Z = L$ is a line. We denote by $\mathcal{O}_{(\mu)} = \mathcal{O}/I^{\mu+1}$ the structure sheaf of the μth infinitesimal nbd. The data of a second-order element of arc crossing L is equivalent to locally giving a section of $\mathcal{O}_{(2)}$ defined up to multiplication by units in $\mathcal{O}^*_{(2)}$.

Explicitly, let $\{U_\alpha\}$ be an open covering of a nbd of L in $\mathbb{P}^2$ s.t in U_α we have holomorphic coordinates (z_α, w_α) with $L \cap U_\alpha$ defined by $w_\alpha = 0$. In $U_\alpha \cap U_\beta$, $z_\alpha = z_\alpha(z_\beta, w_\beta)$ and $w_\alpha = w_\alpha(z_\beta, w_\beta)$, where $w_\beta(z_\beta, 0) = 0$.

┌ Abuse of notations. ∫