The fact that $\bar\partial^* G_{\bar\partial}\eta$ has pure type (p. 9-1) has nothing to with $\partial(\bar\partial^* G_{\bar\partial}\eta) = 0$.

$\partial\bar\partial^* + \bar\partial^*\partial = 0$ by P115. (in the proof of lemma).

$\Rightarrow$ $\partial(\bar\partial^* G_{\bar\partial}\eta) = -\bar\partial^* G_{\bar\partial}(\partial\eta) = 0$ in case. d is choosen., i.e. η is d-exact.

Since the harmonic space for ∂ is the same as the harmonic space for $\bar\partial$ and hence is orthogonal to the range of $\bar\partial^*$, we deduce by the Hodge decomposition for ∂ that

$$\bar\partial^* G_{\bar\partial}\eta = \partial\partial^* G_\partial(\bar\partial^* G_{\bar\partial}\eta).$$

Pf Since $\Delta_\partial = \Delta_{\bar\partial}$, the harmonic space for ∂ is the same as the harmonic space for $\bar\partial$.

$\mathcal{H}_{\bar\partial} \oplus \bar\partial^* D$ i.e $\mathcal{H}_{\bar\partial} \perp \mathrm{Im}\,\bar\partial^*$

$\Rightarrow$ $\mathcal{H}_{\bar\partial} = \mathcal{H}_\partial \perp \mathrm{Im}\,\bar\partial^*$.

$$\bar\partial^* G_{\bar\partial}\eta = \underbrace{\mathcal{H}_{\bar\partial}(\bar\partial^* G_{\bar\partial}\eta)}_{0} + \overset{=0}{\cancel{\bar\partial\bar\partial^*\bar\partial^* G_{\bar\partial}\eta}} + \bar\partial^*\bar\partial\bar\partial^* G_{\bar\partial}\eta$$

$$\Rightarrow \mathcal{H}_{\bar\partial}(\bar\partial^* G_{\bar\partial}\eta) = \mathcal{H}_\partial(\bar\partial^* G_{\bar\partial}\eta) = 0$$

$$\Rightarrow \bar\partial^* G_{\bar\partial}\eta = \partial\partial^* G_\partial\bar\partial^* G_{\bar\partial}\eta + \overbrace{\partial^*(\partial G_\partial\bar\partial^* G_{\bar\partial}\eta)}^{0}$$

$$= \partial\partial^* G_\partial(\bar\partial^* G_{\bar\partial}\eta)$$

By commuting the various operators,

$$\eta = \pm\bar\partial\partial(\partial^*\bar\partial^* G_{\bar\partial}^2\eta),$$