Consider $\overline{q', p_1, \cdots p_{n-1}} = \mathbb{P}^{n-1} = H_\lambda$ for some $\lambda \in \mathbb{P}^1$.
$\Rightarrow q' = q(\lambda)$.

$\Rightarrow$ Every point of C lies on a hyplane H_λ.

Note that, since $\{q', p_1, p_2 \cdots p_{n-1}\}$ is a linearly independent set, it determines a unique H_λ.

We have to make corrections follows:

Every point of C except $p_1, p_2 \cdots p_{n-1}$ will lie on a unique hyperplane H_λ.

Let $\gamma : \mathbb{C} \longrightarrow C$ be an analytic curve s.t $\gamma(0) = p_1$.

Consider $\overline{p_1, p_2, \cdots p_{n-1}, \gamma(t)} = H_t$ which is a hyperplane containing $\overline{p_1, p_2, p_{n-1}}$ in $\mathbb{P}^n$ for sufficiently small t, since $\gamma(t) \notin \{p_1, p_2, \cdots p_{n-1}\}$.

$$\Rightarrow \lim_{t \to 0} \overline{p_1, p_2, \cdots p_{n-1}, \gamma(t)} = \lim_{t \to 0} H_t = H_{\lambda_1} \text{ for some } \lambda_1 \in \mathbb{P}^1.$$

$\Rightarrow q(\lambda_1) = p_1$. Similarly, q is onto.

$q : \mathbb{P}^1 - \{\lambda_1, \cdots \lambda_{n-1}\} \longrightarrow C - \{p_1, p_2 \cdots p_{n-1}\}$ is (see P19) biholomorphic by the implicit function theorem, see P9. $\Rightarrow$ By Riemann's removable singularity theorem q can be extended. Similarly, q^{-1} can be extended.
$\Rightarrow q : \mathbb{P}^1 \longrightarrow C$ is biholomorphic. $\Rightarrow$
$q : \mathbb{P}^1 \to C$ is an isomorphism.