$\mathcal{D}(\psi, \eta) = \langle \varphi, \eta \rangle$ and define $T(\varphi) = \psi$. ⅃⅃

T is self-adjoint, since this is true of I and Δ.

Γ $\langle \varphi, \eta \rangle = \langle (I+\Delta) T\varphi, \eta \rangle \Rightarrow \varphi = (I+\Delta)T\varphi$.

$\Rightarrow \langle T\varphi, \varphi \rangle = \langle T\varphi, (I+\Delta)T\varphi \rangle \geq 0$ since $I+\Delta$ is positive definite.

$\Rightarrow \langle T\varphi, \varphi \rangle \geq 0$ for all φ $\Rightarrow$ By Th 12.32. (Rudin FA)

$T = T^*$ p 313. ⅃⅃

From $2\alpha\beta \leq \epsilon\alpha^2 + \frac{1}{\epsilon}\beta^2$ and

$$\|T\varphi\|_1^2 \leq C\mathcal{D}(T\varphi, T\varphi) = C\langle \varphi, T\varphi \rangle$$

Cauchy-Schwartz inequality

$$\leq C\|\varphi\|_0 \|T\varphi\|_0 \leq 2\epsilon C\|T\varphi\|_0^2 + \frac{2C}{\epsilon}\|\varphi\|_0^2,$$

we deduce that

$$\|T\varphi\|_1^2 \leq C'\|\varphi\|_0^2. \quad (\text{Take } \epsilon \text{ small enough})$$

This says that T is bounded as a map from $\mathcal{H}_0^{p,q}(M)$ to $\mathcal{H}_1^{p,q}(M)$, and by the global Rellich lemma it is compact as an operator on $\mathcal{H}_0^{p,q}(M)$.

Γ $\|T\varphi\|_1^2 - 2\epsilon C\|T\varphi\|_0^2 \leq \frac{2C}{\epsilon}\|\varphi\|_0^2$

$$\| $$

$$\|T\varphi\|_0^2 + \sum_{[\alpha]=1}\|D^\alpha T\varphi\|_0^2 - 2\epsilon C\|T\varphi\|_0^2$$

$$\| $$

$$(1-2\epsilon C)\|T\varphi\|_0^2 + \sum_{[\alpha]=1}\|D^\alpha T\varphi\|_0^2 \geq (1-2\epsilon C)\Big(\|T\varphi\|_0^2 + \sum_{[\alpha]=1}\|D^\alpha T\varphi\|_0^2\Big)$$

$$= (1-2\epsilon C)\|T\varphi\|_1^2$$

⅃⅃