$\Lambda = \overline{p_i, \Lambda \cap \mathbb{P}_i^{n-1}}$.

we see that any $\Lambda \ni p_i$ meets $\overline{V}_4$ in a line if and only if $\Lambda$ contains either of the points $p_{i1}$ or $p_{i2}$.

$\Gamma$ Suppose $\overline{V}_4 \cap \mathbb{P}_i^{n-1} \supset \mathbb{P}^2$. $\Rightarrow$ Consider $\overline{\mathbb{P}^2, p_i}$ which is $\mathbb{P}^3$ since $p_i \notin \mathbb{P}_i^{n-1}$. $\Rightarrow$ $\overline{\mathbb{P}^2, p_i} \subset T_{p_i}(F)$ and $p_i \subset \overline{V}_2 \subset \overline{V}_4$.
$\Rightarrow$ $\overline{V}_4$ & $F$ do not intersect each other transversely.
$\Rightarrow$ Contradiction to the choice of $\overline{V}_\alpha$'s, see P 739.
$\Rightarrow$ $\overline{V}_4 \cap \mathbb{P}_i^{n-1} = \mathbb{P}^1$ $\quad$ $\mathbb{P}_i^{n-1} \cap F = F_i$ is a smooth quadric in $\mathbb{P}_i^{n-1}$ $\Rightarrow$ $F_i$ contains $\mathbb{P}^1$ or meets $\mathbb{P}^1$ in two points. If $F_i \supset \mathbb{P}^1$, then consider $\mathbb{P}^2 = \overline{p_i, \mathbb{P}^1} \subset \overline{V}_4$, and $\mathbb{P}^2 \subset F$. Then again this contradicts to the fact that $\overline{V}_4$ will intersect $F$ in a smooth 2-dimensional quadric $F_2$, and $F_2$ can not contain $\mathbb{P}^\ell$, $\ell > 1$.
If $\Lambda \ni p_i$, then $\Lambda \subset T_{p_i}(F)$.
Since $\mathbb{P}_i^{n-1} \subset T_{p_i}(F)$, $\dim(\Lambda \cap \mathbb{P}_i^{n-1}) \geq k + n - 1 - n = k-1$.
$p_i \notin \mathbb{P}_i^{n-1}$ $\Rightarrow$ $\dim(\Lambda \cap \mathbb{P}_i^{n-1}) = k-1$.
$\Rightarrow$ $\Lambda = \overline{p_i, \Lambda \cap \mathbb{P}_i^{n-1}}$

($\Rightarrow$) If $\Lambda \cap \overline{V}_4$ is a line, since $p_i \in \Lambda \cap \overline{V}_4$, & $\Lambda = \overline{p_i, \Lambda \cap \mathbb{P}_i^{n-1}}$, $\exists$ $p \in \overline{V}_4 \cap \Lambda \cap \mathbb{P}_i^{n-1}$.
$\Rightarrow$ $p \in \mathbb{P}_i^{n-1} \cap \Lambda \subset \mathbb{P}_i^{n-1} \cap F = F_i$ $\Rightarrow$ $p \in \overline{V}_4 \cap F_i = \{p_{i1}, p_{i2}\}$

($\Leftarrow$) Suppose $\Lambda \ni p_{i1}$
$\Rightarrow$ $\overline{p_i p_{i1}} \subset \Lambda$ and since $p_{i1} \in \overline{V}_4$, $\overline{p_i p_{i1}} \subset \overline{V}_4$
$\Rightarrow$ $\Lambda \cap \overline{V}_4 \supset \overline{p_i p_{i1}}$ $\Rightarrow$ By the argument on P 821 note, $\Lambda \cap \overline{V}_4 = \overline{p_i p_{i1}}$. Here $\Lambda$ need not be in $\Sigma_{k,n} \cap \sigma_{n-k, n-k-1}$. $\lrcorner$