$m_1, \cdots m_k$ generate $M \Leftrightarrow$ they generate M_0.

Proof. The implication $\Rightarrow$ is obvious. Conversely, assume that $m_1, \cdots m_k$ generate M_0 and let $S \subset M$ be the submodule of M that they generate.

$\ulcorner$ $M_0 = \{ m_1 + mM, \cdots m_k + mM \}$
$S = \{ m_1, m_2, \cdots m_k \} \subset M$. $\lrcorner$

To show that $S = M$ we set $Q = M/S$ and consider the exact sequence

$$0 \longrightarrow S \longrightarrow M \xrightarrow{\pi} Q \longrightarrow 0.$$

If $m' \in M$, then $m' - s' \in mM$ for some $s' \in S$.

$\ulcorner$ $m' + mM = (a_1 m_1 + mM) + \cdots (a_k m_k + mM) = (a_1 m_1 + \cdots + a_k m_k) + mM$
$\Rightarrow m' - (a_1 m_1 + \cdots + a_k m_k) \in mM$ but $a_1 m_1 + \cdots + a_k m_k \in S$.
$\Rightarrow$ Let $s' = a_1 m_1 + \cdots + a_k m_k$. $\lrcorner$

Consequently, $\pi(m') = \pi(m' - s') \in mQ$ and $Q = mQ$. Then $Q = (0)$ as desired.

$\ulcorner$ $\pi(m' - s') = \pi(m') - \pi(s') = \pi(m')$ since $\pi(s') = s' + S = S$.
$\pi(m' - s') = (m' - s') + S = \sum \underset{m}{\underset{\cap}{m_j}} \underset{M}{\underset{\cap}{M_j}} + S = \sum m_j (M_j + S)$

$\in mQ \Rightarrow \pi(M) = Q \subset mQ \Rightarrow$ Since $mQ \subset Q$,
$Q = mQ$. $\Rightarrow$ By Nakayama's lemma, $Q = (0)$. $\Rightarrow M/S = (0)$.
$\Rightarrow M = S$. $\lrcorner$