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Now, for $\psi \in A_c^{n,n-1}(\bar{W})$, $f^*\psi$ is compactly supported and

$$\bar\partial \sigma_h(\psi) = \int_{\bar W} \sigma_h(w)\, \bar\partial\psi(w)$$
$$= \int_U h(z)\, \bar\partial\, (f^*\psi)(z)$$
$$= -\int_U \bar\partial h(z)\, f^*\psi(z)$$
$$= 0$$

since $h \in \mathcal{O}(U)$. Q.E.D.

$\ulcorner$ Since f is proper, $f^*\psi$ is compactly supported.
See P553 note & refer to P535 note.

$$\bar\partial \sigma_h(\psi) = \int_{\bar W} \sigma_h(w)\, \bar\partial\psi(w)$$
$$= \int_U h(z)\, (f^*\bar\partial\psi)(z) = \int_U h(z)\, \bar\partial\psi\circ f\,(z)$$
$$= \int_U h(z)\, (\bar\partial(f^*\psi))(z), \text{ since } f \text{ is holomorphic.}$$
$$(\bar\partial\cdot f^* = f^*\bar\partial)$$
$$= -\int_U \bar\partial h(z)\, (f^*\psi)(z) \text{ by Stokes' theorem}$$
$$= 0 \qquad (\text{integration by parts})$$

$\lrcorner$

If we apply the lemma to the power sums $\sum_\nu h(z_\nu(w))^k$, then we deduce that any symmetric function — such as

$$h(z_1(w))\cdot \cdots \cdot h(z_d(w))$$