Thus r and $\beta + hf$ both are zero on the inverse image of any component of the zero locus of γ other than $\pi(W)$; but r and $\beta + hf$ are relatively prime and so have no common components.

Γ

$$\begin{array}{ccc} \mathbb{C}^n & \supset & W \\ \pi \downarrow & & \downarrow \\ \mathbb{C}^{n-1} & \supset & \pi(W) \end{array}$$

$\Rightarrow$ Let $(\gamma = 0) - \pi(W) = K$.
For any $z \in K$, $r = 0$ & $\beta + hf = 0$ on $\pi^{-1}(z)$.

Since f & g are relatively prime in $\mathcal{O}_{n-1}[z_n]$, if r and $\beta + hf$ are not relatively prime in $\mathcal{O}_{n-1}[z_n]$, by the resultant $\gamma = rf + (\beta + hf)g$,
$\gamma \in \mathcal{O}_{n-1}[z_n]$ i.e γ contains z_n^{ℓ} term. $\Rightarrow$ $*$.

Let $(\gamma = 0) - \pi(W) = K$, as above. $\Rightarrow$ K is open in $(\gamma=0)$ since $\pi(W)$ is closed ($\because$ $\pi : (f=0) \longrightarrow \mathbb{C}^{n-1}_{(loc)}$ is proper by Weierstrass Preparation Theorem. i.e $\pi : (f=0) \longrightarrow \mathbb{C}^{n-1}(loc)$ is a finite sheeted branched cover onto a nbd of 0.
Similarly we have the same thing for $(g=0)$. $\Rightarrow$
$\pi : (f=0) \cap (g=0) \longrightarrow \pi(W)$ is closed map.)
$\Rightarrow$ For each $z' \in K$, $\exists$ open set $U \ni z'$ s.t

$U \cap K = (\ell = 0)$, where ℓ is holomorphic on U, since $(\gamma = 0)$ is analytic variety in some open set $-$ $\pi(W)$.