Finally, we denote by $I(f)=\{f_1\cdots f_n\}$. the ideal generated by the f_i's in the ring of germs of holomorphic functions around 0. Then

$$\mathrm{Res}_{\{0\}}\,\omega = 0 \quad \text{in case } g\in I(f).$$

To prove this, it suffices by linearity to consider the case $g=hf_1$. But then

$$\omega = \frac{h(z)\ dz_1\wedge\cdots\wedge dz_n}{f_2(z)\ \cdots\ f_n(z)}$$

is holomorphic in the larger open set $U_{\{1\}^c} = U-(D_2+\cdots+D_i+\cdots+D_n)$.

F $I(f)$ is the ideal generated by $f_1\cdots f_n$.
$\Rightarrow$ $I(f)=\{\sum h_if_i \mid h_i\in\mathcal{O}_n\}$
$\Rightarrow$ By linearity, and since we may assume hf_1 without loss of generality, we have only to prove for hf_1. Strange expression, somtimes we need to not write down

If Γ_i is the chain

$$\Gamma_i=\{z: |f_j(z)|=\epsilon_j \text{ for } j\neq i,\ |f_i(z)|\le\epsilon_i\},$$

then $\Gamma_1\subset U_{\{1\}^c}$ and $\partial\Gamma_1=\pm\Gamma$. Hence $\int_\Gamma\omega=\pm\int_{\Gamma_1}d\omega=0$ by Stokes' theorem

F For example, $U_{\{1\}^c}=U-D_2$ $\quad \Gamma_1=\{z\in U \mid |f_2(z)|=\epsilon_2,\ |f_1(z)|\le\epsilon_1\}$
$z\in\Gamma_1 \Rightarrow f_2(z)\neq0 \Rightarrow z\in U-\{f_2(z)=0\}$
$\Rightarrow \Gamma_1\subset U_{\{1\}^c}$. $\quad \Gamma_1\cong\{w\in U: |w_j|=\epsilon_j \text{ for } j\neq i,\ |w_i|\le\epsilon_i\}$