$$F(\lambda_1, \cdots \lambda_n) = f \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

then F is a symmetric holomorphic function in $\lambda_1, \cdots \lambda_n$.

Γ Since $f(gAg^{-1}) = f(A)$, and

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ a & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix},$$

F is symmetric in $\lambda_1 \cdots \lambda_n$. ⌟

Write $F(\lambda_1, \cdots \lambda_n) = G(\sigma_1, \cdots \sigma_n)$, where $\sigma_1, \sigma_2, \cdots \sigma_n$ are the elementary symmetric polynomials in the λ_i; the equality

$$f(A) = G(P^1(A), \cdots P^n(A))$$

then holds throughout the connected and dense open set of semisimple (i.e., diagonalizable) matrices in GL_n, hence in all of M_n.

Γ F can be expressed in terms of power series and each part of homogeneous deg n is symmetric in $\lambda_1, \cdots \lambda_n$. $\Rightarrow$ By the fundamental theorem, each homogeneous part of deg n is expressed as a polynomial in $\sigma_1, \sigma_2 \cdots \sigma_n$, where $\sigma_1, \sigma_2, \cdots \sigma_n$ are the elementary symmetric polynomials.

Let K be the set of all diagonalizable matrices in GL_n. Given $A \in K$, $\exists\ g \in GL_n$ s.t