$$\begin{tikzcd}[column sep=small]
H^{n-1}(U^*, Z_{\bar\partial}^{n,0}) \arrow[r, "\cong"] & H^{n-2}(U^*, Z_{\bar\partial}^{n,1}) \arrow[r, "\cong"] & H^{n-3}(U^*, Z_{\bar\partial}^{n,2}) \arrow[r, "\cong"] & \cdots \arrow[r, "\cong"] & H^1(U^*, Z_{\bar\partial}^{n,n-2}) \\
\omega_{n-1} \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \omega_{n-2} \arrow[u, phantom, "\in"] \arrow[r, mapsto] & \omega_{n-3} \arrow[u, phantom, "\in"] & & \omega_1 \arrow[u, phantom, "\in"] \; \arrow[u, shift left=8] \\
\left(\frac{1}{2\pi\sqrt{-1}}\right)^n \omega \arrow[u, phantom, "\parallel"] & & & \eta_\omega = \omega_0 \arrow[ur, shift left] \arrow[ur, leftarrow, shift right] & \in H^0(U^*, Z_{\bar\partial}^{n,n-1}) = Z_{\bar\partial}^{n,n-1}(U^*) \\
& & & & H^0(U^*, \mathcal{A}^{n,n-2}) \arrow[u] \\
& & & & \mathcal{A}^{n,n-2}(U^*) \arrow[u, phantom, "\parallel"]
\end{tikzcd}$$

$$\Rightarrow \quad \eta_\omega = \omega_{0,i} \quad \text{on } U_i.$$

$$\sum_{i\in I} \int_{\Gamma_i} \omega_{0,i} = \sum_{i\in I = \{1,\dots,n\}} \int_{\Gamma_i} \eta_\omega = \int_{\partial \Gamma_0} \eta_\omega, \quad \Gamma_0 = \{z : |f_i(z)| \le \epsilon\}$$

$$= \int_{S^{2n-1}} \eta_\omega, \qquad \partial\Gamma_0 \equiv S^{2n-1} = \partial(\{z : |f_i(z)| \le \epsilon\}) . ?$$

We observe that this lemma does not use the assumption that ω is meromorphic in U with polar divisor D. Only $\omega \in H^0(U-D, \Omega^n)$ is required, so that ω could have a higher order pole or even an essential singularity along D. In case ω is meromorphic with polar divisor D, we may find a distinguished representative for the Dolbeault class η_ω as follows. Set

$$\rho_i = \frac{|f_i|^2}{|f_1|^2 + \cdots + |f_n|^2}$$

and observe that

$$\rho_i \text{ is } C^\infty \text{ in } U^*,$$
$$\sum_i \rho_i \equiv 1$$

and

$$\operatorname{supp}(\rho_i) \subset U_i.$$