We note first that, inasmuch as L is part of the branch locus of the map, every line through p_0 in the plane h meets S in exactly one more point and is tangent to S there.

$\ulcorner$ Let l be a line through p_0 in h. $\Rightarrow l \cap L \neq \phi \Rightarrow$ If we let $p = l \cap L$, then since $p \in F$, $r : \tilde{C}_h \longrightarrow \mathbb{P}^1$ is branched at p.
$\Rightarrow \#(l \cap S - p_0) = 1 \Rightarrow$
Since h contains p_0 and other double points only on L, l must be tangent to S.
Here the case $l = \overline{p_0 p_i}$ is included, since 'tangent' means multiplicities ≥ 2. $\lrcorner$

" Mak-up for the observation above.

Let h be a hyperplane through p_0 with 5 double points other than p_0. Say $p_1, p_2 \in h$.

$$\begin{tikzcd} r : \tilde{S} \arrow[rr] \arrow[dr] & & \mathbb{P}^2 \\ & S \arrow[ur] & \end{tikzcd}$$

$r(p_1) \in h \cap \mathbb{P}^2 = L$, and $r(p_2) \in L \Rightarrow L$ contains one of six lines forming F, say $L = L_1$
$\Rightarrow h = \overline{p_0, L_1} \Rightarrow$ Thus there are 6 hyperplane containing six of the p_i's. "