

$$\begin{aligned} \langle D'\eta, \tau \rangle &= \langle \partial \eta_\alpha \otimes e_\alpha + \sum \eta_\beta \wedge \theta_{\beta\alpha} \otimes e_\alpha, \tau_\alpha \otimes e_\alpha \rangle \\ &= \langle \partial \eta_\alpha, \tau_\alpha \rangle + \langle \sum_\beta \eta_\beta \wedge \theta_{\beta\alpha}, \tau_\alpha \rangle \end{aligned}$$

$$\Rightarrow \langle D'\eta, \tau \rangle$$

$$\begin{aligned} &= \int \langle \partial \eta_\alpha, \tau_\alpha \rangle + \langle \sum_\beta \eta_\beta \wedge \theta_{\beta\alpha}, \tau_\alpha \rangle \\ &= \int \sum_\alpha \partial \eta_\alpha \wedge * \tau_\alpha + \sum_\beta \eta_\beta \wedge \theta_{\beta\alpha} \wedge * \tau_\alpha \end{aligned}$$

$$\langle \eta, D'^*\tau \rangle = \langle \eta_\alpha \otimes e_\alpha, \partial^* \tau_\alpha \otimes e_\alpha + \theta^* \tau \rangle$$

$$= \int \sum_\alpha \langle \eta_\alpha, \partial^* \tau_\alpha \rangle + \langle \eta_\alpha, \theta^* \tau \rangle$$

$$= \int \sum_\alpha \eta_\alpha \wedge * \partial^* \tau_\alpha + \int \sum_\alpha \langle \eta_\alpha, \theta^* \tau \rangle$$

$$\Rightarrow \int \sum_\alpha \partial \eta_\alpha \wedge * \tau_\alpha - \int \sum_\alpha \eta_\alpha \wedge * \partial^* \tau_\alpha$$

$$= \int \sum_\alpha \partial (\eta_\alpha \wedge * \tau_\alpha) = \int_M d(\sum_\alpha \eta_\alpha \wedge * \tau_\alpha) = 0$$

Define $\theta^*: A^{p,q}(E)(U) \longrightarrow A^{p-1,q}(E)(U)$ by

$$\eta_\alpha \wedge \theta_{\alpha\beta} \wedge * \tau_\beta = \langle \eta_\alpha \otimes e_\alpha, \theta^* \tau \rangle \mathbb{I},$$

where \mathbb{I} is the volume form.