But $x_{n+\alpha}$ is a well-defined function on $V/\mathbb{Z}\{\lambda_1, \cdots \lambda_n\}$, so $[d x_{n+\alpha}] = 0 \in H^1_{DR}(V/\mathbb{Z}\{\lambda_1, \cdots \lambda_n\})$.

$\ulcorner$ $x_{n+\alpha}(a_\alpha \lambda_\alpha + \mathbb{Z}\{\lambda_1, \cdots \lambda_n\}) = a_{n+\alpha}$. $\Rightarrow$ $x_{n+\alpha}$ is well-defined on $V/\mathbb{Z}\{\lambda_1, \cdots \lambda_n\}$ globally.

$$\begin{array}{ccc} A^0(V/\mathbb{Z}\{\lambda_1, \cdots \lambda_n\}) & \xrightarrow{d} & A^1(V/\mathbb{Z}\{\lambda_1, \cdots \lambda_n\}) \\ \cup & & \cup \\ x_{n+\alpha} & \longmapsto & d x_{n+\alpha}. \end{array}$$

$\Rightarrow$ $H^1_{DR}(V/\mathbb{Z}\{\lambda_i\}) = \dfrac{\ker d}{\operatorname{Im} d} \ni [d x_{n+\alpha}] = 0$ since

$d x_{n+\alpha} \in \operatorname{Im} d$.

$\lrcorner$

Thus
$$c_1(\pi_1^* L) = \pi_1^*(c_1(L)) = 0,$$
and consequently $\pi_1^* L$ is trivial.

$\ulcorner$

$$c_1(L) = \sum \delta_\alpha\, dx_\alpha \wedge dx_{n+\alpha}$$
$$\pi_1^* c_1(L) = c_1(\pi_1^* L) = \pi_1^*\left(\sum \delta_\alpha\, dx_\alpha \wedge dx_{n+\alpha}\right)$$
$$= \sum \delta_\alpha\, \pi_1^* dx_\alpha \wedge \pi_1^* dx_{n+\alpha} = \sum \delta_\alpha\, \pi_1^* dx_\alpha \wedge d\, x_{n+\alpha} \circ \pi_1$$
since $x_{n+\alpha} \circ \pi_1 = x_{n+\alpha}$ globally.
$$= d\left(\sum \delta_\alpha\, \pi_1^* dx_\alpha \wedge dx_{n+\alpha}\right) = 0 \in H^2_{DR}(V/\Lambda'), \quad \Lambda' = \mathbb{Z}\{\lambda_1, \cdots \lambda_n\}.$$
$\Rightarrow$ Since any line bundle on $V/\mathbb{Z}\{\lambda_1, \cdots \lambda_n\}$ is determined by