$$\iota(v)\,\Theta = \iota(v)\left(-\sum \frac{\partial \Gamma^{j}_{ik}}{\partial \bar z_l}\, \underbrace{dz_k \wedge d\bar z_l}_{}\right)\frac{\partial}{\partial z_j}\otimes dz_i$$

$\nearrow$ (1,1) type indicator

$$= -\sum \frac{\partial \Gamma^{j}_{ik}}{\partial \bar z_l}\, v^k\, d\bar z_l\, \frac{\partial}{\partial z_j}\otimes dz_i$$

$$= -\sum_{i,j,k}\left(\frac{\partial \Gamma^{j}_{ik}}{\partial \bar z_l}\, v^k \frac{\partial}{\partial z_j}\otimes dz_i\right) d\bar z_l .$$

$$\Rightarrow \quad \iota(v)\,\Theta = -\bar\partial E. \qquad \lrcorner$$

Now, consider Θ, E, and $\bar\partial E = -\iota(v)\,\Theta$ again as matrix-valued 2-, 0-, and 1-forms, respectively; if P is any invariant polynomial of degree n on $GL(n)$ and $\tilde P$ its polarization, set

$$P_r(E,\Theta) = \binom{n}{r}\,\tilde P(\underbrace{E,\cdots E}_{n-r},\ \Theta,\cdots\ \Theta) \in A^{r,r}(M^*).$$

Since $\bar\partial\Theta = 0$ and $\bar\partial E = -\iota(v)\,\Theta$,

$$\bar\partial P_r(E,\Theta) = \binom{n}{r}\sum_{i=1}^{n-r} \tilde P(E,\cdots\ \iota(v)\cdot\Theta,\cdots E,\ \Theta,\cdots\Theta)$$

$$= \iota(v)\cdot P_{r+1}(E,\Theta).$$

$\ulcorner$ Since $\Theta = \sum \Theta_{ij}\,\frac{\partial}{\partial z_j}\otimes dz_i$, and $\frac{\partial}{\partial z_j}\otimes dz_i$'s are holomorphic,

$$\bar\partial\Theta = \sum \bar\partial\Theta_{ij}\otimes\frac{\partial}{\partial z_j}\otimes dz_i$$

$$= \sum \bar\partial(\bar\partial\Theta_{ij})\otimes\frac{\partial}{\partial z_j}\otimes dz_i = 0.$$

$$\bar\partial P_r(E,\Theta) = \binom{n}{r}\sum_{i=1}^{n-r}\tilde P(\overbrace{E,\cdots\ \ \overset{i}{\bar\partial E},\cdots E}^{},\ \Theta,\cdots\Theta)$$