But now on V we have

$$\varphi^d - \pi^*\psi_1 \cdot \varphi^{d-1} + \pi^*\psi_2 \cdot \varphi^{d-2} - \cdots + (-1)^d \pi^*\psi \equiv 0,$$

i.e., every meromorphic function $\varphi \in \mathcal{M}(V)$ satisfies a polynomial relation of degree d over $\pi^* K(\mathbb{P}^k)$.

⌈ We have only to show that for each $q \in V$,

$$\varphi(q)^d - \pi^*\psi_1(q) \cdot \varphi(q)^{d-1} + \pi^*\psi_2(q) \cdot \varphi(q)^{d-2} - \cdots + (-1)^d \pi^*\psi(q) = 0$$

$$\Rightarrow \varphi(q)^d - \psi_1(\pi(q))\, \varphi(q)^{d-1} + \psi_2(\pi(q)) \cdot \varphi(q)^{d-2} + \cdots + (-1)^d \psi(\pi(q)) = 0$$

$\Rightarrow$ let $\pi(q) = p$,

$$\prod_{q' \in \pi^{-1}(p)} (\varphi(q) - \varphi(q')) = \varphi(q)^d - \psi_1(\pi(q))\, \varphi(q)^{d-1} + \cdots + (-1)^d \psi(p) = 0$$

since $q \in \pi^{-1}(p)$. ⌋

By the primitive element theorem, then, the field extension $\mathcal{M}(V) \supset \pi^* K(\mathbb{P}^k)$ is finite of degree at most d.

⌈ We have only to show the following lemma.

lemma: $F \supset K$ algebraic extension of K. and for every $\alpha \in F$, $[K(\alpha), K] \le d$. Then show that $[F : K] < \infty$.

pf) Suppose $[F : K] = \infty$.

$\Rightarrow \exists$ a countable basis elements $\alpha_1, \alpha_2, \alpha_3 \cdots$ s.t. $[K(\alpha_1, \alpha_2, \cdots \alpha_n \cdots) : K] = \infty$.