Pf L, given a trivialization $\{(\varphi_\alpha, U_\alpha)\}$ and another trivialization $\{(\psi_\alpha, U_\alpha)\}$,

$$\begin{tikzcd}
\pi^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "\psi_\alpha"'] & U_\alpha \times \mathbb{C} \arrow[dl, "f_\alpha"] \\
U_\alpha \times \mathbb{C} \arrow[ur, phantom, "\circlearrowright"] &
\end{tikzcd}$$

$$f_\alpha(x, z) = (x, \underset{\substack{\uparrow \\ \mathbb{C}^*}}{f_\alpha(x)\cdot z}) = f_\alpha(x)(x, z)$$ since $\psi_\alpha \circ \varphi_\alpha^{-1} : U_\alpha \times \mathbb{C} \to U_\alpha \times \mathbb{C}$ is holomorphically linear.

$\Rightarrow$ $\cancel{\psi_\alpha \circ \varphi_\alpha^{-1}(x,z) = f_\alpha(x,z)}$ $\Rightarrow$ $\psi_\alpha \circ \varphi_\alpha^{-1}(x, z) = f_\alpha(x, -) = f_\alpha(x)\cdot z)$ multiplication.

$\Rightarrow$ $\cancel{\psi_\alpha \circ \varphi_\alpha^{-1}(\varphi_\alpha(v)) = f_\alpha(\varphi_\alpha(v))}$ where $\varphi_\alpha(v) = (x, z)$.

$\psi_\alpha(v) = f_\alpha(\varphi_\alpha(v)) = (x, f_\alpha$

$$\psi_\alpha \circ \varphi_\alpha^{-1} = f_\alpha \quad \Rightarrow \quad \psi_\alpha = f_\alpha \circ \varphi_\alpha = f_\alpha(x)\,\varphi_\alpha$$

$$\psi_\alpha(v) = f_\alpha(\varphi_\alpha(v)) = f_\alpha(x)\cdot \varphi_\alpha(v)$$

$\Rightarrow$ $\psi_\alpha = f_\alpha \cdot \varphi_\alpha$ where $f_\alpha \in \mathcal{O}^*(U_\alpha)$.

The set of line bundles on M is just $H^1(M, \mathcal{O}^*)$.

Given a line bundle L on M, we have a trivialization $\{(\varphi_\alpha, U_\alpha)\}$ which defines transition functions $\{g_{\alpha\beta}\}$, which is a 1-cocycle in $C^1(\underline{U}, \mathcal{O}^*)$, where $\underline{U} = \{U_\alpha\}$.
$\Rightarrow$ $\{g_{\alpha\beta}\} \in H^1(\underline{U}, \mathcal{O}^*) \longrightarrow H^1(M, \mathcal{O}^*)$.

Conversely, given an element $l \in H^1(M, \mathcal{O}^*)$, $\exists$ an open cover $\{V_\alpha\}$ s.t. $l \in H^1(\underline{V}, \mathcal{O}^*)$. $\Rightarrow$ l defines a line bundle. $\rfloor\rfloor$