pf) Given a bounded sequence $\{u_k\}$ in H_s, we want to find a convergent subsequence in H_r. Since, for all k, we have

$$\sum (1+\|\xi\|^2)^r |u_{k,\xi}|^2 \leq \sum (1+\|\xi\|^2)^s |u_{k,\xi}|^2 < C,$$

, for fixed ξ, the sequence $\{(1+\|\xi\|^2)^{r/2} u_{k,\xi}\}_{k=1}^{\infty}$ is bounded and hence has a Cauchy sequence.
(we need convergent sequence).

Then by the standard diagonalization, we can find a subsequence $\{u_k\}$ s.t $\{(1+\|\xi\|^2)^{r/2} u_{k,\xi}\}_{k=1}^{\infty}$ is convergent
(we will use the same notation) (convergent) (Cauchy)
for every $\xi \in \mathbb{Z}^n$.

Now we seperate the terms with small ξ, of which there are only a finite number, from those with large ξ where the factor $(1+\|\xi\|^2)^r$ will help: given $\epsilon > 0$, choose R and m such that

$$\frac{4C}{(1+\|\xi\|^2)^{s-r}} < \frac{\epsilon}{2} \quad \text{for } \|\xi\| \geq R.$$

$$\sum_{\|\xi\| \leq R} (1+\|\xi\|^2)^r |u_{k,\xi} - u_{\ell,\xi}|^2 < \frac{\epsilon}{2} \qquad \text{for } k, \ell \geq m.$$

Then
$$\|u_k - u_\ell\|_r^2 = \sum_{\|\xi\| \leq R} (1+\|\xi\|^2)^r |u_{k,\xi} - u_{\ell,\xi}|^2$$
$$+ \sum_{\|\xi\| > R} \frac{(1+\|\xi\|^2)^s}{(1+\|\xi\|^2)^{s-r}} |u_{k,\xi} - u_{\ell,\xi}|^2 < \frac{\epsilon}{2} + \sum_{\|\xi\| > R} \frac{(1+\|\xi\|^2)^s}{(1+\|\xi\|^2)^{s-r}} |u_{k,\xi} - u_{\ell,\xi}|^2$$
$$< \frac{\epsilon}{2} + 2 \sum_{\|\xi\| > R} \frac{(1+\|\xi\|^2)^s |u_{k,\xi}|^2}{(1+\|\xi\|^2)^{s-r}} + 2 \sum_{\|\xi\| > R} \frac{(1+\|\xi\|^2)^s |u_{\ell,\xi}|^2}{(1+\|\xi\|^2)^{s-r}}$$
$$\leq \frac{\epsilon}{2} + 2 \sum \frac{(1+\|\xi\|^2)^s |u_{k,\xi}|^2}{(1+R^2)^{s-r}} + 2 \sum_{\|\xi\| > R} \frac{(1+\|\xi\|^2)^s |u_{\ell,\xi}|^2}{(1+R^2)^{s-r}} \leq \frac{2C}{(1+R^2)^{s-r}} \times 2$$
$$+ \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$