We call such an S a horizontal slice for $f: M \to \mathbb{C}^N$. It is clear from the implicit function theorem that there is a nbd W of p_0 such that $f(W) = f(S \cap W)$. It follows that $f(M) = f(S)$, since M is irreducible.

Γ By the assumption on P395, $\exists$ a coordinate chart U s.t $p_0 \in U$

$$\begin{array}{ccccc} \mathbb{C}^n & \xleftarrow{\varphi} & U & \xrightarrow{f} & \mathbb{C}^N \\ \uparrow & & \cup & & \\ \mathbb{C}^k & \longleftarrow & U \cap S & & \end{array}$$

$f|_*$ has rank k at p_0

$\Rightarrow$ $f|_*$ has rank k on an open set $W \ni p_0$.

$\Rightarrow$ On $W \ni p_0$,

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \\ \frac{\partial f_k}{\partial z_1} & \cdots & \frac{\partial f_k}{\partial z_n} \\ \vdots & & \\ \frac{\partial f_N}{\partial z_1} & \cdots & \frac{\partial f_N}{\partial z_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}, & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \\ \frac{\partial f_k}{\partial z_1} & \cdots & \frac{\partial f_k}{\partial z_n} \\ \vdots & & \vdots \end{pmatrix} \text{at } (z_1, \cdots z_k, 0 \cdots 0)$$

$$\Rightarrow \quad \frac{\partial f_1}{\partial z_{k+1}} = 0 = \cdots = \frac{\partial f_1}{\partial z_n} \quad \text{at } (z_1, \cdots z_k, 0, \cdots 0).$$

$$\vdots$$

$$\frac{\partial f_N}{\partial z_{k+1}} = 0 = \cdots = \frac{\partial f_N}{\partial z_n} \quad \text{at } (z_1 \cdots z_k\ 0 \cdots 0)$$

since $$\frac{\partial f_i(z_1, \cdots z_k, 0 \cdots 0)}{\partial z_l} = \left. \frac{\partial f_i(z_1 \cdots z_n)}{\partial z_l} \right|_{(z_1, \cdots z_k, 0, \cdots 0)}.$$