Lemma. If $T \in \mathcal{D}(\mathbb{R}^n)$ satisfies

$$\Delta T = \eta \in C_c^\infty(\mathbb{R}^n),$$

then $T = T_\psi$ for some $\psi \in C^\infty(\mathbb{R}^n)$ such that $\Delta\psi = \eta$.

[ $\Delta T = \eta$ means that $\eta = T_\eta$ for $\eta \in C_c^\infty(\mathbb{R}^n)$ ]

Proof. We will write down an explicit solution $\rho \in C^\infty(\mathbb{R}^n)$ to the equation

$$\Delta \rho = \eta,$$

using the classical Green's function

$$G(x,y) = \begin{cases} \dfrac{1}{\|x-y\|^{n-2}}, & n \geq 3 \\ \log \|x-y\|. & n = 2. \end{cases}$$

Then

$$\Delta(T - T_\rho) = 0,$$

and this lemma follows from the preceding one.

We shall assume that $n \geq 3$, the case $n=2$ being essentially the same.

Define

$$\rho(x) = C_n \int_{y \in \mathbb{R}^n} \frac{\eta(y)\, dy}{\|x-y\|^{n-2}}$$

$$= \pm C_n \int_{u \in \mathbb{R}^n} \frac{\eta(x-u)\, du}{\|u\|^{n-2}}$$

where the equality follows from the change of variables $y = x-u$.