Now we have an exact sheaf sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{A} \xrightarrow{\exp} \mathcal{A}^* \longrightarrow 0$$

and since the long exact sequence in Čech cohomology is functorial, the inclusion maps $\mathcal{O} \to \mathcal{A}$ and $\mathcal{O}^* \to \mathcal{A}^*$ give a commutative diagram

$$\begin{array}{ccccc} H^1(M, \mathcal{A}) & \longrightarrow & H^1(M, \mathcal{A}^*) & \xrightarrow{\delta'} & H^2(M, \mathbb{Z}) \\ \uparrow & & \uparrow & & \uparrow \\ H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{O}^*) & \xrightarrow{\delta} & H^2(M, \mathbb{Z}). \end{array}$$

with both rows exact. Thus we can define the Chern class $c_1(L)$ of a C^∞ line bundle to be $\delta'(L)$, and this definition agrees with the one above for holomorphic bundles.

But in the upper row we have $H^1(M, \mathcal{A}) = 0$, since the sheaf is fine; the conclusion is that a complex line bundle is determined up to C^∞ isomorphism by its Chern class.

Recall now that for any vector bundle $E \to M$ of rank k and any connection D on E, the curvature operator D^2 is represented, in terms of a trivialization φ_α of E over U_α, by a $k \times k$ matrix Θ_α of 2-forms; if φ_β is another trivialization, we have

$$\Theta_\alpha = g_{\alpha\beta}\, \Theta_\beta \cdot g_{\alpha\beta}^{-1},$$

where $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL_k$ is the transition function relative to φ_α & φ_β. In particular, if E is a line bundle, since $GL_1 = \mathbb{C}^*$ is commutative, $\Theta = \Theta_\alpha = \Theta_\beta$ is a closed, globally defined differential form of degree 2, called the curvature form of E.