Date . .

Consider a holomorphic section $s = a_0 s_0 + a_1 s_1 + \cdots + a_N a_N$ of L over M, where $\{s_0, \cdots s_N\}$ forms a basis for E.

Let $K = \{s=0\}$. $\Rightarrow p \in K \Rightarrow s(p) = a_0 s_0(p) + \cdots + a_N s_N(p) = 0$ -- ①

$\iota_E(p) = [s_{0\alpha}(p), \cdots s_{N,\alpha}(p)]$

$$\begin{tikzcd} L|_{U_\alpha} \arrow[r, "\varphi_\alpha"] & U_\alpha \times \mathbb{C} \\ s_i(p) \arrow[r, leftrightarrow] \arrow[d] & (p, s_{i\alpha}(p)) \arrow[dl] \\ U_\alpha & \end{tikzcd}$$

Since φ_α is linear, $a_0 s_{0\alpha}(p) + \cdots + a_N s_{N\alpha}(p) = 0$ --- ②
$\Rightarrow \iota_E(p) \in \{\tau = 0\} \Rightarrow \tau \circ \iota_E(p) = 0$. $p \in \{\sigma = 0\}$
$\Rightarrow \{s=0\} \subset \{\sigma = 0\}$.

Conversely, $p \in \{\sigma = 0\} \Rightarrow \iota_E(p) \in \{\tau = 0\}$.
$\Rightarrow \iota_E(p) = [s_{0,\alpha}(p), \cdots s_{N\alpha}(p)]$ satisfies $a_0 s_{0\alpha}(p) + \cdots + a_N s_{N\alpha}(p) = 0$
$\Rightarrow a_0 s_0(p) + \cdots + a_N s_N(p) = 0 \Rightarrow p \in \{s=0\}$.

Thus $\{s=0\} = \{\sigma = 0\}$, and since multiplicities of zeros of ① & ② are same, $(s=0) = (\sigma = 0)$.
$\Rightarrow$ By P136, $\iota_E^*(H) = L$.

Moreover, any section $s = \sum a_i s_i \in E$ is the pullback of the section $\sum a_i Z_i$ of H on $\mathbb{P}^N$; i.e.,

$$E = \iota_E^*(H^0(\mathbb{P}^N, \mathcal{O}(H))) \subset H^0(M, \mathcal{O}(L)).$$

We already proved above.

Thus $\iota_E : M \longrightarrow \mathbb{P}^N$ determines both the line bundle L

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