$$= \sum (-1)^j \sigma_{a_j+d-j} \cdot \sigma_{a_1 \cdots a_{j-1}, a_{j+1}-1 \cdots a_d-1} = \sigma_{a_1 \cdots a_d}.$$

and the formula is proved. Q.E.D.

Γ Expand by cofactors along the last column.

$$\begin{vmatrix} \sigma_{a_1} & \cdots & \sigma_{a_1+d-1} \\ \vdots & & \sigma_{a_2+d-2} \\ & & \\ \sigma_{a_d-d+1} & & \sigma_{a_d} \end{vmatrix} = (-1)^{d-1} \sigma_{a_1+d-1} \cdot \begin{vmatrix} \sigma_{a_2-1} & \cdots & \sigma_{a_2+d-3} \\ \sigma_{a_3-2} & \cdots & \\ & & \\ \sigma_{a_d-d+1} & & \end{vmatrix}$$

$$+ (-1)^d \sigma_{a_2+d-2} \cdot \begin{vmatrix} \sigma_{a_1} & \cdots & \sigma_{a_1+d-2} \\ \sigma_{a_3-2} & \cdots & \sigma_{a_3+d-3} \\ \vdots & & \end{vmatrix} + (-1)^{d+1} \sigma_{a_3+d-3} \cdot \begin{vmatrix} \sigma_{a_1} & \cdots & \sigma_{a_1+d-2} \\ \sigma_{a_2-1}, & \cdots & \sigma_{a_2+d-3} \\ \sigma_{a_4-3} & \cdots & \end{vmatrix}$$

by the assumption.

$$\cdots = (-1)^{d-1} \sigma_{a_1+d-1} \cdot \sigma_{a_2-1, a_3-1, \cdots a_d-1} + (-1)^d \sigma_{a_2+d-2} \cdot$$

$$\sigma_{a_1, a_3-1, \cdots} + (-1)^{d+1} \sigma_{a_3+d-3} \cdot \sigma_{a_1, a_2, a_4-1, \cdots}$$

$$= \sum \sigma_{a_j+d-j} \cdot \sigma_{a_1, \cdots a_{j-1}, a_{j+1}-1, \cdots a_d-1} (-1)^{d+j-2}$$

$$= (-1)^d \sum (-1)^j \sigma_{a_j+d-j} \cdot \sigma_{a_1 \cdots a_{j-1}\, a_{j+1}-1, \cdots a_d-1}$$

$$= (-1)^d (-1)^d \sigma_{a_1 \cdots a_d} = \sigma_{a_1 \cdots a_d}.$$