$$\Rightarrow J(F) = \begin{pmatrix} \frac{\partial f_1}{\partial z_1}, & \frac{\partial f_1}{\partial z_2}, & \frac{\partial f_1}{\partial z_3}, & 0, & 0 \\ \frac{\partial f_2}{\partial z_1}, & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial z_3}, & 0, & 0 \\ \frac{\partial f_3}{\partial z_1}, & \frac{\partial f_3}{\partial z_2}, & \frac{\partial f_3}{\partial z_3}, & 1, & 0 \\ \frac{\partial f_4}{\partial z_1}, & \frac{\partial f_4}{\partial z_2}, & \frac{\partial f_4}{\partial z_3}, & 0, & 1 \end{pmatrix}$$

$$\Rightarrow \left| \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & 0, & 0 \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & 0 & 0 \\ \frac{\partial f_3}{\partial z_1} & \frac{\partial f_3}{\partial z_2} & 1, & 0 \\ \frac{\partial f_4}{\partial z_1}, & \frac{\partial f_4}{\partial z_2}, & 0, & 1 \end{pmatrix} \right| \neq 0 .$$

$F \mid \mathbb{C}^2 \times \{0\} \times \mathbb{C}^2 \longrightarrow \mathbb{C}^4$ is locally biholomorphic.

$\Rightarrow$ By using the argument above, we have an open set W in $\mathbb{C}^5$ s.t $F(W) = F(W \cap \mathbb{C}^4)$.

In special, $F(\{(z_1, z_2, 0, 0 0)\}) =$ ... is not clear as this pen. I am going to complete the argument this time

Point is this: Inverse function theorem implies the implicit function theorem, and the implicit function theorem depends on the "sort of" initial conditions as the solutions of ordinary differential equations depend on the initial conditions smoothly.

We will explain this by giving an example.

$$f : \mathbb{C}^3 \longrightarrow \mathbb{C}^4$$

$$(z_1, z_2, z_3) \longmapsto (f_1, f_2, f_3, f_4). \quad \det\left(\frac{\partial f_i}{\partial z_j}\right) \neq 0 \text{ at the origin. } \quad 1 \le i, j \le 2.$$

and $\operatorname{rank} f = 2$.