*

$$\mathcal{H}^{p,q} \oplus \bar\partial A^{p,q-1} \oplus \bar\partial^* A^{p,q+1} \xrightarrow{\ \Delta\ } \mathcal{H}^{p,q} \oplus \bar\partial A^{p,q-1} \oplus \bar\partial^* A^{p,q+1}$$

$$\begin{tikzcd} \Delta : \bar\partial A^{p,q-1} \oplus \bar\partial^* A^{p,q+1} \arrow[r] & \bar\partial A^{p,q-1} \oplus \bar\partial^* A^{p,q+1} \arrow[l, bend left, "G"] \end{tikzcd}$$ isomorphic bijective

$$\Delta \circ G = G \circ \Delta = I$$

*

So ~~in fact~~ in effect what we shall be doing is trying to solve the Laplace equation on a compact manifold. The idea is to first solve this equation in the weak sense – i.e in the Hilber space ~~decomposition~~ completion $\mathcal{L}^{p,q}(M)$ of $A^{p,q}(M)$ to find a ψ s.t $(\psi, \Delta\varphi) = (\eta, \varphi)$ for all $\varphi \in A^{p,q}(M)$ – and then to prove that this ψ is in fact C^∞. (by integration by parts, $\langle \Delta\psi, \varphi\rangle = \langle \psi, \Delta\varphi\rangle = \langle \eta, \varphi\rangle \Rightarrow \Delta\psi = \eta$).
The first step is pretty much formal Hilbert-space theory, and the second, – usually called the regularity theorem – is at least a local problem, since φ may be written as a sum of forms with compact support in coordinate patches.

Proof of The Hodge Theorem I : Local Theory

The proof of the Hodge Theorem given here uses elementary Hilber-space techniques. We are looking for the elements of smallest norm in the affine subspace $\psi + \bar\partial A^{p,q-1}(M) \subset A^{p,q}(M)$. Clearly such an element can be found in the closure of $\psi + \bar\partial A^{p,q-1}(M)$ in the completion $\mathcal{L}^{p,q}(M)$ of the pre-Hilbert space $A^{p,q}(M)$, simply by orthogonal projection. The problem then is to show that the element found in this way in fact lies in $A^{p,q}(M)$. We start by discussing functions on the torus.