expressed as a linear combination of σ_1, σ_2 over V.
$\Rightarrow$ Since $\sigma, \sigma_1, \sigma_2$ are holomorphic,
by the identity theorem, σ is expressed as a linear combination of σ_1, σ_2 globally.

Question ; Even if $f=0$, $\frac{a_{11}}{a_{12}} = \frac{a_{21}}{a_{22}}$ need not be constant over U ?

$\frac{\sigma_1}{\sigma_2} = h$, meromorphic function on U

Correction : If f is zero over U,
, for each x, $\sigma_1(x) = h(x)\,\sigma_2(x)$.
We can not say that σ_1, σ_2 are linearly dependent over U.
If $a_{12}a_{23} - a_{22}a_{13} = 0$ over U,
for each x, $\sigma_2(x) = k(x)\,\sigma_3(x)$
$\Rightarrow \{\sigma_1(x), \sigma_2(x), \sigma_3(x)\}$ can not span E_x.
Thus we can conclude that
$$\begin{pmatrix} a_{11}(x), & a_{21}(x) \\ a_{12}(x), & a_{22}(x) \\ a_{13}(x), & a_{23}(x) \end{pmatrix}$$
has to have rank 2 for some $x \in U$.

In general,
$$\begin{aligned} \sigma_1 &= a_{11}e_1 + \cdots + a_{k1}e_k \\ \sigma_2 &= a_{12}e_1 + \cdots + a_{k2}e_k \\ &\ \vdots \\ \sigma_n &= a_{1n}e_1 + \cdots + a_{kn}e_k . \end{aligned}$$
Some $k \times k$ minor matrix of (a_{ij}) must have k rank for each x.