$\deg(L\otimes K_S^*) = \langle c_1(L\otimes K_S^*), [S]\rangle = \langle c_1(L) + c_1(K_S^*), [S]\rangle = \langle c_1(L), [S]\rangle - \langle c_1(K_S), [S]\rangle$
$= \deg L - \deg K_S > 0 \Rightarrow L\otimes K_S^*$ is positive

$H^1(S, \mathcal{O}(L)) = H^1(S, \Omega^1(L\otimes K_S^*)) = H^1(S, \Omega^1(L\otimes(T^*S)^*)) = 0$ since $1+1 > \dim S = 1$, by Kodaira vanishing theorem, P154. ⌟

Alternatively, this fact follows from Kodaira-Serre duality:
$\deg L > \deg K_S \Rightarrow \deg(K_S\otimes L^*) < 0$
$\Rightarrow H^1(S, \mathcal{O}(L)) \cong H^0(S, \mathcal{O}(K_S\otimes L^*)) = 0.$

⌜ By P153, $H^1(S, \mathcal{O}(L)) \cong H^0(S, \mathcal{O}(L^*\otimes K_S))^*$
$\cong H^0(S, \mathcal{O}(L^*\otimes K_S)) = 0$ by the fact above that $\deg L < 0 \Rightarrow H^0(S, \mathcal{O}(L)) = 0$. ⌟

Now for any $p\in S$, consider the exact sequence

$$0 \longrightarrow \mathcal{O}(L-p) \longrightarrow \mathcal{O}(L) \xrightarrow{r_p} L_p \to 0.$$

⌜ By P139, we have the exact sequence

$$0 \to \mathcal{O}_S(L\otimes[-p]) \longrightarrow \mathcal{O}_S(L) \xrightarrow{r} \mathcal{O}_p(L|_p) \to 0$$

$L|_p = L_p \Rightarrow \mathcal{O}_p(L|_p) =$ the sheaf of holomorphic sections over p which correspond to points on L_p. $\Rightarrow \mathcal{O}_p(L|_p) = L_p$.
$\Rightarrow$ We get

$$0 \to \mathcal{O}(L-p) \longrightarrow \mathcal{O}(L) \xrightarrow{r_p} L_p \longrightarrow 0.$$ ⌟