in case $f_1(p) \neq 0$ on U.

$\Rightarrow$

$$\iota_{\tilde{C}*}(p) = \begin{pmatrix} \dfrac{\partial \frac{f_2}{f_1}}{\partial x}, & \dfrac{\partial \frac{f_2}{f_1}}{\partial y} \\ \dfrac{\partial \frac{f_3}{f_1}}{\partial x}, & \dfrac{\partial \frac{f_3}{f_1}}{\partial y} \\ \dfrac{\partial \frac{f_4}{f_1}}{\partial x}, & \dfrac{\partial \frac{f_4}{f_1}}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2}{f_1^2}, & \dfrac{\frac{\partial f_2}{\partial y} f_1 - \frac{\partial f_1}{\partial y} f_2}{f_1^2} \\ \dfrac{\frac{\partial f_3}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_3}{f_1^2}, & \dfrac{\frac{\partial f_3}{\partial y} f_1 - \frac{\partial f_1}{\partial y} f_3}{f_1^2} \\ \dfrac{\frac{\partial f_4}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_4}{f_1^2}, & \dfrac{\frac{\partial f_4}{\partial y} f_1 - \frac{\partial f_1}{\partial y} f_4}{f_1^2} \end{pmatrix}$$

$\Rightarrow$ $\iota_{\tilde{C}*}(p)$ does not have rank 2.

$\Rightarrow \exists \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \neq 0$ s.t $\iota_{\tilde{C}*}(p) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$.

$\Rightarrow$

$$a_1 \left(\frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 \right) + a_2 \left(\frac{\partial f_2}{\partial y} f_1 - \frac{\partial f_1}{\partial y} f_2 \right) = 0$$

$$\Rightarrow a_1 \frac{\partial f_2}{\partial x} f_1 + a_2 \frac{\partial f_2}{\partial y} f_1 - \left(\frac{\partial f_1}{\partial x} a_1 f_2 + \frac{\partial f_1}{\partial y} f_2 a_2 \right) = 0$$

$$\Rightarrow \left(a_1 \frac{\partial f_2}{\partial x} + a_2 \frac{\partial f_2}{\partial y} \right) f_1 = \left(\frac{\partial f_1}{\partial x} a_1 + \frac{\partial f_1}{\partial y} a_2 \right) f_2$$

Similarly, we have

$$f_1 \cdot \left(\frac{\partial f_3}{\partial x} a_1 + \frac{\partial f_3}{\partial y} a_2 \right) = f_3 \cdot \left(\frac{\partial f_1}{\partial x} a_1 + \frac{\partial f_1}{\partial y} a_2 \right)$$

$$f_1 \cdot \left(\frac{\partial f_4}{\partial x} a_1 + \frac{\partial f_4}{\partial y} a_2 \right) = f_4 \cdot \left(\frac{\partial f_1}{\partial x} a_1 + \frac{\partial f_1}{\partial y} a_2 \right)$$

$$\Rightarrow (f_1, f_2, f_3, f_4) = \beta \left(\frac{\partial f_1}{\partial x} a_1 + \frac{\partial f_1}{\partial y} a_2, \frac{\partial f_2}{\partial x} a_1 + \frac{\partial f_2}{\partial y} a_2, \frac{\partial f_3}{\partial x} a_1 + \frac{\partial f_3}{\partial y} a_2, \frac{\partial f_4}{\partial x} a_1 + \frac{\partial f_4}{\partial y} a_2 \right).$$

$\Rightarrow$ This contradicts to ⊛. This implies that $\iota_{\tilde{C}*}(p)$