We can make the arguments much simpler

$V = \{ f_1 = \cdots = f_k = 0 \}$

Suppose f_1 & f_2 are relatively prime. $\Rightarrow$ Done.

If not, $f_1 = h g_1$, $f_2 = h g_2$.

$\Rightarrow \quad \alpha g_1 + \beta g_2 = \gamma \in \mathcal{O}_{n-1}$

$\Rightarrow \quad V \subset \{ h\gamma = f_3 \cdots = f_k = 0 \}$

$\Rightarrow$ We can use the induction on k. //

If we let $g(z)$ be the greatest common divisor of the f_i's, then we can write

$$V = \{ g(z) = 0 \} \cup \left\{ \frac{f_1(z)}{g(z)} = \cdots = \frac{f_k(z)}{g(z)} = 0 \right\}.$$

Since V is irreducible at 0 and since the locus $\{ f_i(z)/g(z) = 0, \text{ all } i \}$ can not contain $\{ g(z) = 0 \}$, we must have

$$V = \{ g(z) = 0 \},$$

i.e., V is an analytic hypersurface near 0.

Γ $\{ f_i(z)/g(z) = 0, \text{ all } i \} \neq \phi \Rightarrow V$ is reducible at 0.

$\Rightarrow \{ f_i(z)/g(z) = 0, \text{ all } i \} = \phi \Rightarrow V = \{ g(z) = 0 \}$. //

The results 1, 2 and 3 above, together with our basic picture of an analytic hypersurface, give us a picture of the local behavior of those analytic varieties cut out locally by one or two holo-