$L(\eta) = \eta \wedge \omega$ and let $\Lambda = L^*: A^{p,q}(M) \longrightarrow A^{p-1,q-1}(M)$ be its adjoint. Now, for general M, there are no nonobvious relationships among these various operators. If we assume that the metric on M is Kähler, however, we get a host of identities relating them, called the Hodge identities. Indeed, the Kähler condition is exactly that which insures a strong interplay between the real potential theory associated to the Riemannian metric and the underlying complex structure. The basic identity, from which all the others will easily follow, is

$$(*) \quad [\Lambda, d] = -4\pi d^{c*},$$

where $[A, B]$ denotes the commutator $AB - BA$; or equivalently

$$[L, d^*] = 4\pi d^c.$$

pf) By decomposition into type, this identity is equivalent to

$$[\Lambda, \bar\partial] = -i\partial^* \quad \text{and} \quad [\Lambda, \partial] = i\bar\partial^*$$

$$[\Lambda, d] = \Lambda d - d\Lambda = \Lambda(\partial + \bar\partial) - (\partial + \bar\partial)\Lambda$$

$$= \Lambda\partial - \partial\Lambda \quad - \bar\partial\Lambda + \Lambda\bar\partial = -\frac{i}{4\pi}(\bar\partial^* - \partial^*) \times (-4\pi)$$

$$\partial : (p,q) \longmapsto (p+1, q) \xrightarrow{\Lambda} (p, q-1)$$

$$\Lambda : (p,q) \longmapsto (p-1, q-1) \xrightarrow{\partial} (p, q-1)$$

$$\bar\partial : (p,q) \longmapsto (p, q+1) \xrightarrow{\Lambda} (p-1, q)$$

$$\Lambda : (p,q) \longmapsto (p-1, q-1) \xrightarrow{\bar\partial} (p-1, q)$$

$$\bar\partial^* : (p,q) \longmapsto (p, q-1) \qquad \partial^* : (p,q) \longmapsto (p-1, q)$$

$$\Rightarrow \quad \Lambda\partial - \partial\Lambda = i\bar\partial^* \qquad -\bar\partial\Lambda + \Lambda\bar\partial = -i\partial^*$$