The result we are aiming for is the

Kodaira Embedding Theorem. Let M be a compect complex manifold and $L \to M$ a positive line bundle. Then there exists k_0 s.t for $k \geq k_0$, the map

$$\iota_{L^k} : M \longrightarrow \mathbb{P}^N$$

is well-defined and is an embedding of M.

Let us consider how one might go about proving this. The first thing to do is to fit the maps (*) and (**) into exact sequences and try to use our vanishing theorems directly. To this end, let $\mathcal{J}_{x,y}(L)$ denote the sheaf of sections of L vanishing at x and y, and $\mathcal{J}_x^2(L)$ the sheaf of sections of L vanishing to order 2 at x, i.e., sections s of $\mathcal{J}_x(L)$ s.t $d_x(s) = 0$.

$\lceil$ $ds_\alpha = ds_\beta \cdot g_{\alpha\beta} \Rightarrow (ds_\alpha)(x) = ds_\beta(x) \cdot g_{\alpha\beta}(x)$
$\Rightarrow ds_\alpha(x) = 0 \iff ds_\beta(x) = 0$ since $g_{\alpha\beta}(x) \neq 0$.
$\Rightarrow d_x s$ is well-defined. and $d_x(s) = 0$ is well-defined, too. $\rfloor$

We have exact sheaf sequences

$$0 \longrightarrow \mathcal{J}_{x,y}(L) \longrightarrow \mathcal{O}(L) \xrightarrow{r_{x,y}} L_x \oplus L_y \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{J}_x^2(L) \longrightarrow \mathcal{J}_x(L) \xrightarrow{d_x} T_x^{*\prime} \otimes L_x \longrightarrow 0$$

so that to show that the maps (*) and (**) are surj-