$$\to H_q(V) \longrightarrow H_q(M, \mathbb{Z}) \to H_q(M, V) \cong H_q(S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_\ell}, \mathbb{Z}) \underset{\text{if } q \le n-1}{=} 0$$

$$H_{q+1}(S^{n_1} \vee S^{n_2} \vee \cdots \vee S^{n_\ell}, \mathbb{Z}) \quad \text{if} \quad q \le n-2$$

$\Rightarrow$ If $q \le 2+n-4$, $\quad H_q(V, \mathbb{Z}) \cong H_q(M, \mathbb{Z})$.
If $q = n-1$, $\quad H_q(V, \mathbb{Z}) \longrightarrow H_q(M, \mathbb{Z})$ is onto.

For the homotopy group,

$$\pi_{q+1}(M, V) \to \pi_q(V) \longrightarrow \pi_q(M) \longrightarrow \pi_q(M, V)$$

$\Rightarrow$ By p294. Maunder Th 7.4.15. a) $\pi_q(K, L) = 0$ for $1 \le q \le n-1$, if L is a subcomplex that contains K^{n-1}.

$\Rightarrow$ Since V contains $(n-1)$-skeleton, $\pi_q(M, V) = 0$ for $1 \le q \le n-1$.

$\Rightarrow$ If $q \le n-2$, $\pi_{q+1}(M, V) = \pi_q(M, V) = 0$
$\Rightarrow$ $\pi_q(V) \cong \pi_q(M)$. and $\pi_{n-1}(V) \longrightarrow \pi_{n-1}(M)$ is onto.

When $n=1$, the theorem doesn't say anything. However, when $n=2$ – i.e. M is a (connected and compact) complex surface – and $V \subset M$ is a Riemann surface embedded as a positive divisor, then the Lefschetz theorem gives

$$H_0(V, \mathbb{Z}) \cong H_0(M, \mathbb{Z}) = \mathbb{Z}$$
$$H_1(V, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \longrightarrow 0,$$

i.e. all of the first homology of the 4-manifold M lies