a Riemann surface S serves to define a holomorphic map

$$f : z \longmapsto [1, f_1(z), \cdots f_n(z)]$$

from S to $\mathbb{P}^n$: while f is defined a priori only away from the poles of the functions f_i, at any point $p = (z=0) \in S$ we may set

$$m = \max\{-\mathrm{ord}_p(f_i)\}_i,$$

"An extension is a local problem"

and the map

$$\tilde{f} : z \longmapsto [z^m, z^m f_1(z), \cdots, z^m f_n(z)]$$

extends f over p. The fact that we are using here to extend f is simply that the point is a divisor on the Riemann surface S; i.e., it is defined by a single function z, and any function vanishing at p must be divisible by z. This, of couse, fails in higher codimension, and so we may expect that a general rational map will not be everywhere defined. The simplest case is the rational map

$$f : \mathbb{C}^2 \longrightarrow \mathbb{P}^1$$

given by the single meromorphic function $f(x, y) = y/x$, i.e., by

$$f(x, y) = [1, y/x] = [x, y] :$$

f is well-defined and holomorphic away from the origin $(0, 0) \in \mathbb{C}^2$, but can not be extended to a map on all of $\mathbb{C}^2$.

Suppose $\exists$ an extension of f. $\Rightarrow$ $x = y$ and $x \to 0$,