$$\Rightarrow \quad f(t,z) = \int a(t,z)\,dt + \sqrt{-1}\int b(t,z)\,dt \text{ is holph.}$$

More precisely, let $v(z) = v(z)\frac{\partial}{\partial z} = (p(x,y) + \sqrt{-1}\, q(x,y))\frac{\partial}{\partial z}$ which corresponds to

$$p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}.$$

where $\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y}$ & $\frac{\partial q}{\partial x} = -\frac{\partial p}{\partial y}$.

$$\frac{\partial}{\partial t} f(t,z) = p(f(t,x,y))\frac{\partial f(t,x,y)}{\partial x} + q(f(t,x,y))\frac{\partial f(t,)}{\partial y}.$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}f\right) = \frac{\partial p(f())}{\partial x}\frac{\partial f}{\partial x} + p\circ f\cdot\frac{\partial^2 f}{\partial x^2} + \frac{\partial q\circ f}{\partial x}\frac{\partial f}{\partial y} + q\frac{\partial^2 f}{\partial y\partial x}$$

$$= \left(\frac{\partial p}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial p}{\partial y}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial x} + p\circ f\frac{\partial^2 f}{\partial x^2} + \left(\cancel{\frac{\partial q\circ f}{\partial x}}\frac{\partial f}{\partial x} + \frac{\partial q}{\partial y}\frac{\partial f}{\partial x}\right)\frac{\partial f}{\partial y} + q\frac{\partial^2 f}{\partial y\partial x} \quad \text{—— ①}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}f\right) = \left(\frac{\partial p}{\partial x}\frac{\partial f}{\partial y} + \cancel{\frac{\partial p}{\partial y}\frac{\partial f}{\partial y}}\right)\frac{\partial f}{\partial x} + p\circ f\frac{\partial^2 f}{\partial x\partial y} + \left(\frac{\partial q}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial q}{\partial y}\frac{\partial f}{\partial y}\right)\frac{\partial f}{\partial y} + q\frac{\partial^2 f}{\partial y^2} \quad \text{—— ②}$$

① + ② = No! Completely WRONG! ⇑

For the notation $f_t = \exp(tv)$, see P 80, Differentiable Manifolds by Matsushima. By the theorem 2 on 121, the f_t is holomorphic, for each t. ⅃

Moreover, if $z_1, \cdots z_n$ are local coordinates around a zero p of v and