By Cramer's rule this implies

$$\Delta \cdot m_j = 0,$$

where

$$\Delta = \det(\delta_{ij} - a_{ij}) \in 1 + m .$$

Thus Δ is a unit, and so $I = 0$. Q.E.D.

Pf

$$\det\begin{pmatrix} b_{11}, & \cdots & 0 \\ b_{21} & & 0 \\ \vdots & & \vdots \\ b_{k1}, & & 0 \end{pmatrix} = \det\begin{pmatrix} b_{11}, & \cdots & \sum b_{1\ell} m_\ell & \cdots \\ \vdots & & \vdots & \\ b_{k1}, & & \sum b_{k\ell} m_\ell & \end{pmatrix}$$

(column j)

$$= \det\begin{pmatrix} b_{11} & b_{11}\cdot m_1 \\ \vdots & \vdots \\ b_{k1} & b_{k1}\cdot m_1 \end{pmatrix} + \cdots + \det\begin{pmatrix} b_{11} & \cdots & b_{1j} m_j \\ \vdots & & \vdots \\ b_{k1}, & \cdots & b_{kj} m_j \end{pmatrix} + \cdots$$

$$= m_j \det\begin{pmatrix} b_{11} & b_{1j} \\ \vdots & \vdots \\ b_{k1} & b_{kj} \end{pmatrix} = \Delta\cdot m_j ,$$ where $b_{ij} = \delta_{ij} - a_{ij}$.

$\Delta = \det(\delta_{ij} - a_{ij}) = 1 + \sum a_{ij}\square \in 1 + m$, since m is an ideal. $\Rightarrow$ Since $\Delta(0) = 1(0) + x(0) = 1$, Δ is a unit, where $x \in m$, $x(0) = 0$. $\Rightarrow m_j = 0 \Rightarrow$ Every element in M is zero. $\Rightarrow M = (0)$.

The Nakayama lemma is most useful in the following form: