Now $\Delta_d = (\partial+\bar\partial)(\partial^*+\bar\partial^*) + (\partial^*+\bar\partial^*)(\partial+\bar\partial)$

$= (\partial\partial^* + \partial^*\partial) + (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) + (\partial\bar\partial^* + \bar\partial\partial^* + \cancel{\partial^*\bar\partial + \bar\partial^*\partial})$

$= \partial\partial^* + \partial^*\partial + \bar\partial\partial^* + \bar\partial^*\bar\partial = \Delta_\partial + \Delta_{\bar\partial}$.

So we have to show $\Delta_\partial = \Delta_{\bar\partial}$.

For this, $-i\,\Delta_\partial = \partial(\Lambda\bar\partial - \bar\partial\Lambda) + (\Lambda\bar\partial - \bar\partial\Lambda)\partial$
$= \partial\Lambda\bar\partial - \partial\bar\partial\Lambda + \Lambda\bar\partial\partial - \bar\partial\Lambda\partial$

and consequently

$i\,\Delta_{\bar\partial} = \bar\partial(\Lambda\partial - \partial\Lambda) + (\Lambda\partial - \partial\Lambda)\bar\partial$
$= \bar\partial\Lambda\partial - \bar\partial\partial\Lambda + \Lambda\partial\bar\partial - \partial\Lambda\bar\partial$
$= \bar\partial\Lambda\partial + \partial\bar\partial\Lambda - \Lambda\bar\partial\partial - \partial\Lambda\bar\partial$
$= i\,\Delta_\partial$ since $\bar\partial\partial = -\partial\bar\partial$. Q.E.D.

As an immediate corollary, we see that Δ_d preserves bidegree i.e

$$[\Delta_d, \Pi^{p,q}] = 0.$$

⟦ Since $\Delta_{\bar\partial} : A^{p,q} \longrightarrow A^{p,q}$, and $\Delta_{\bar\partial} = \frac{1}{2}\Delta_d$, $\Delta_d\,\Pi^{p,q} = \Pi^{p,q}\Delta_d$. ⟧

There are two main applications of these identities, the Hodge decomposition and the Lefschetz decomposition and theorem. We do Hodge first:

Set
$$H^{p,q}(M) = \frac{Z_d^{p,q}(M)}{dA^*(M) \cap Z_d^{p,q}(M)}$$

$$\mathcal{H}_d^{p,q}(M) = \{\eta \in A^{p,q}(M) ; \Delta_d\eta = 0\}$$

$$\mathcal{H}_d^{r}(M) = \{\eta \in A^{r}(M) ; \Delta_d\eta = 0\}.$$

Note that the first group is intrinsically defined by the complex structure, while the latter two depend on the