Date

$\Rightarrow$ $i(L)$ has degree $c_1(L) = 3$. $\Rightarrow$ $i(L)$ is a cubic curve in $\mathbb{P}^N$.

But $H^0(S, \mathcal{O}(L))$ corresponds to meromorphic functions on S holomorphic on $S - \{p\}$ and of order ≥ -3 at p; since any such function is uniquely determined by its principal part (implying $a_{-3} \neq 0$)

$$\frac{a_{-3}}{z^3} + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z} + a_0 + \cdots \quad \text{at } p,$$

and since there can not exist a meromorphic function with only a single pole at p — as noted before, such a function would give a 1-1 map of S to $\mathbb{P}^1$ — we see that $h^0(S, \mathcal{O}(L)) \leq 3$, hence $h^0(S, \mathcal{O}(L)) = 3$ and we are done.

$\ulcorner$ Given a section $\sigma \in H^0(S, \mathcal{O}(L))$, consider σ/s_0.

$\Rightarrow (s_0 = 0) = 3p$ $\Rightarrow$ σ/s_0 is a meromorphic function with a pole of order ≥ -3 at p and with less than 3 zeros.

If f, g are meromorphic functions s.t their principal parts are the same up to constant, then f/g is again a meromorphic function on S

① order $= -3$ ($a_{-3} \neq 0$)

f/g is holomorphic on S $\Rightarrow$ f/g = constant

② order $= -2$ ($a_{-3} = 0$ & $a_{-2} \neq 0$)

f/g is holomorphic on $S - \{\text{point}\}$ $\Rightarrow$ By the argument