Since the global sections of $[-E]$ over E represent hyperplanes in $\mathbb{P}(T_x'(M))$, these hyperplanes correspond to the linear functionals on the holomorphic tangent space $T_x'(M)$.

On the other hand, given a function f on $\mathcal{U}$ vanishing at x, the function $\pi^*f \in \mathcal{O}(\tilde{\mathcal{U}})$ vanishes along E and so can be considered as a section of $[-E]$ over $\tilde{\mathcal{U}}$.

Let s_0 be the section s.t $(s_0=0)=E$.
$\Rightarrow$ $\pi^*f \otimes s_0^{-1}$ is a section of $[-E]$ over $\tilde{\mathcal{U}}$.

See P139.
$\Rightarrow$ We have the following exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_{\tilde{\mathcal{U}}}(\tilde{\mathcal{U}}\times\mathbb{C}\otimes[-E]) & \xrightarrow{\otimes s_0} & \mathcal{O}_{\tilde{\mathcal{U}}}(\tilde{\mathcal{U}}\times\mathbb{C}) & \longrightarrow & \mathcal{O}_E(E\times\mathbb{C}) & \to 0 \\ & & & \Psi & & & \\ & & & \pi^*f & \longmapsto & f & \end{array}$$

By explicit computation we check that for any $f\in \mathcal{J}_x(\mathcal{U})$ the restriction to E of the section $\pi^*f \in \mathcal{O}(-E)(\tilde{\mathcal{U}})$ corresponds, via the identification $(**)$, to the differential $df(x)$ of f at x, i.e., the diagram

$$\begin{tikzcd} H^0(\tilde{\mathcal{U}}, \mathcal{O}(-E)) \arrow[r, "r_E"] & H^0(E, \mathcal{O}(-E)) \arrow[d, equal] \\ H^0(\mathcal{U}, \mathcal{J}_x) \arrow[u] \arrow[r, "d_x"] & T_x^{*\prime}(\mathcal{U}) \end{tikzcd}$$

commutes.