The $\varphi_i(z_1)$ are holomorphic and bounded in $0<|z_1|\leq \delta$, and hence they extend to holomorphic functions on the full disc.

$\Gamma(h=0) = D$. Since D does not contain the line $\{z_1 = 0\}$, h is not identically zero on the z_2-axis.
$\Rightarrow$ By the Weierstrass Preparation Theorem,

$$h(z_1, z_2) = z_2^d \cdot g(z_1, z_2). \quad \text{where} \quad g_1(0,0) \neq 0.$$

Wrong since h is not defined at $(0,0)$.
For each z_1 with $0<|z_1|\leq \delta$, we have d roots for $h(z_1, z_2) = 0$, not necessarily distinct. Let $z_{21}(z_1), \cdots z_{2d}(z_1)$ be the d-roots of $h(z_1, z_2) = 0$, for each z_1.

$$\Rightarrow \quad \sigma_1(z_1) = z_{21}(z_1) + \cdots + z_{2d}(z_1)$$

$$\vdots$$

$$\sigma_d(z_1) = z_{21}(z_1)\, z_{22}(z_1) \cdots z_{2d}(z_1).$$

Consider $g(z_1, z_2) = \prod_{\nu=1}^{d} (z_2 - z_{2\nu}(z_1))$

$$= z_2^d - \sigma_1(z_1) z_2 + \cdots \pm \sigma_d(z_1).$$

is analytic since $\sigma_i(z_1)$'s can be expressed in the power sums $\sum z_{2\nu}(z_1)^i$ and each $\forall \sum z_{2\nu}(z_1)^i =$

$$\varphi_i(z_1) = \frac{1}{2\pi\sqrt{-1}} \int_{|z_2|=\epsilon} z_2^i \, \frac{dh(z_1, z_2)}{h(z_1, z_2)} \quad \text{is analytic.}$$

See P8. The above argument is just a kind of practice.

$$|\varphi_i(z)| \leq \frac{1}{2\pi} \int_{|z_2|=\epsilon} \epsilon^i \left| \frac{dh(z_1, z_2)}{h(z_1, z_2)} \right| < \infty.$$