η is in the log complex in case this is true of φ.

Write $(z_1, \cdots, z_n) = (u_1, \cdots, u_k, v_1, \cdots v_{n-k}) = (u, v)$, so that $P^*(k, n)$ is given by

$$\{(u, v): \quad 0 < |u_i| < 1, \quad |v_j| < 1\}$$

and the divisor D by $u_1 \cdots u_k = 0$. We first eliminate the v's from the picture. Following the procedure in the proof of the $\bar{\partial}$-Poincaré lemma in Section 2 of Chapter 0, we suppose that $\varphi \equiv 0$ $(du, dv_1, \cdots dv_\ell)$ and write

$$\varphi = \varphi' + \varphi'' \wedge dv_\ell,$$

where $\varphi', \varphi'' \equiv 0$ $(du, dv_1, \cdots dv_{\ell-1})$. Then $d\varphi = 0$ $\Rightarrow$ $(\partial\varphi'/\partial v_j) = (\partial\varphi''/\partial v_j) = 0$ for $j > \ell$, where if $\alpha = \sum \alpha_I \, dx_I$ is a differential form,

$$\frac{\partial \alpha}{\partial x_j} = \sum_I \frac{\partial \alpha_I}{\partial x_j} dx_I .$$

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$$\varphi = \varphi' + \varphi'' \wedge dv_\ell.$$

$$d\varphi = 0 \quad \Rightarrow \quad d\varphi' + d\varphi'' \wedge dv_\ell = 0$$

$$\Rightarrow \text{ If } j > \ell, \qquad d\varphi' = \pm \boxed{\cdots} \, dv_j + \cdots$$

$$= dv_j \wedge \frac{\partial\varphi'}{\partial v_j} + \cdots \qquad = dv_j \wedge \frac{\partial \varphi'}{\partial v_j} + \sum_{i \neq j} dv_i \wedge \frac{\partial\varphi'}{\partial v_i}$$

and

$$d\varphi'' \wedge dv_\ell = \quad dv_j \wedge \frac{\partial \varphi''}{\partial v_j} \wedge dv_\ell + \sum_{j' \neq j} \cdots$$

$$dv_{j'} \wedge \frac{\partial\varphi''}{\partial v_{j'}} \wedge dv_\ell$$