be a Hermitian metric on the complex manifold M. We say that ds^2 is Kähler if its associated $(1,1)$-form

$$\omega = \frac{i}{2}\sum \varphi_j \wedge \overline{\varphi_j} \qquad \text{is } d\text{-closed.}$$

In section 5 above we showed that there was a unique matrix ψ of 1-forms satisfying

$$\psi_{ij} + \overline{\psi_{ji}} = 0, \quad d\varphi_i = \sum \psi_{ij}\wedge\varphi_j + \tau_i.$$

where τ_i is of type $(2,0)$; there we said that the metric was Kähler if the torsion $\tau = 0$. We now show that these conditions are equivalent. Write.

$$\frac{2}{i}\, d\omega = \sum d\varphi_i \wedge \overline{\varphi_i} - \sum \varphi_i \wedge d\overline{\varphi_i}$$
$$= \sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i} - \sum \varphi_i \wedge \overline{\psi_{ij}} \wedge \overline{\varphi_j} + \sum \tau_i \wedge \overline{\varphi_i}$$
$$- \sum \varphi_i \wedge \overline{\tau_i}.$$

We have

$$\sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i} - \sum \varphi_i\wedge\overline{\psi_{ij}}\wedge\overline{\varphi_j} = \sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i} +$$
$$\sum \varphi_i \wedge \psi_{ji}\wedge\overline{\varphi_j} = 0$$

$$\Big[\!\Big[\sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i} + \sum \varphi_i\wedge\psi_{ji}\wedge\overline{\varphi_j} = \sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i}$$
$$+ \sum (-1)\psi_{ji}\wedge\varphi_i\wedge\overline{\varphi_j} = \sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i} - \sum \psi_{ij}\wedge\varphi_j\wedge\overline{\varphi_i} = 0 \Big]\!\Big]$$

and so $\frac{2}{i}\, d\omega = \sum \tau_i \wedge \overline{\varphi_i} - \sum \varphi_i \wedge \overline{\tau_i}$. But τ_i is of type $(2,0)$, and the $\overline{\varphi_i}$ are pointwise linearly independent $(0,1)$-forms, which implies that $d\omega \Leftrightarrow \tau = 0$.

Another interpretation of the Kähler condition that gives some geometric insight is this: We say a metric ds^2 on M osculates to order k to the Euclidean metric on $\mathbb{C}^n$ if for every point $z_0 \in M$, we can find a holomorphic coordinate system (z) in a nbd of z_0 for which