proof.) Let $|s|^2$ be a metric on L with curvature form Θ. If $\varphi : L_U \longrightarrow U \times \mathbb{C}$ is a trivialization of L over an open set U, s a section of L over U and s_U the corresponding holomorphic function, then

$$|s|^2 = h(z) \cdot |s_U|^2. \quad \text{for some positive function } h(z).$$

Pf

$$\begin{tikzcd}
L_U \arrow[rr, "\varphi"] \arrow[dr] & & U \times \mathbb{C} \arrow[dl] \\
& U \arrow[ul, bend left, dashed, "s"] \arrow[ul, bend left=60, dashed, "e"] \arrow[ur, bend right, dashed, "s_U"'] \arrow[ur, bend right=60, "e_U"'] &
\end{tikzcd}$$

$$s_U = f\, e_U.$$

$$\varphi \circ s = (x, s_U)$$

$$\langle \varphi\circ s, \varphi\circ s \rangle = |s_U|^2$$

$\Rightarrow \quad s = f\, \varphi^{-1} e_U$ where e_U is the corresponding holomorphic function to e which is the unit section over U

$$|s|^2 = |f|^2\, |\varphi^{-1} e_U|^2$$

$\Rightarrow \quad s = f\, e \quad \Rightarrow \quad |s| = |f| \quad \Rightarrow$ Since $s_U = f e_U$,

$$|s_U| = |f|\, |e_U| \quad \Rightarrow \quad |f| = \frac{|s_U|}{|e_U|}, \quad |e_U| \neq 0.$$

$$\Rightarrow \quad |s| = \frac{|s_U|}{|e_U|} \quad \Rightarrow \quad |s|^2 = \frac{|s_U|^2}{|e_U|^2} = h(z)\, |s_U|^2.$$

The curvature form and Chern class are given by

$$\Theta = -\partial\bar\partial \log h(z),$$

$$c_1(L) = \left[\frac{i}{2\pi} \Theta \right] \in H^2_{DR}(M).$$