$Q v_2 = \lambda_2 v_2 \Rightarrow \mathbb{C}^n = \langle v_1 \rangle \oplus \langle v_2 \rangle \oplus v_2^{\perp}$

Continue this procedure. $\Rightarrow \mathbb{C}^n = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle$.

$Q v_i = \lambda_i v_i$, and ${}^t v_i v_i \neq 0$, ${}^t v_i v_j = 0$.

$\Rightarrow Q(v_i, v_j) = {}^t v_i Q v_j = \lambda_j {}^t v_i v_j = \lambda_i {}^t v_i v_j$

$\Rightarrow \lambda_i = \lambda_j$.

Note: ① Since Q is nonsingular, every eigenvalue of Q is nonzero.

② $Q : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a map defined by $Q(X) = QX$.

$\Rightarrow \mathbb{C}^n = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_r}$, where V_{λ_i} is the eigenspace of Q with eigenvalue λ_i.

③ $\lambda_i \neq \lambda_j$ $v_i \in V_{\lambda_i}$ and $v_j \in V_{\lambda_j}$.

$\Rightarrow {}^t v_i Q v_j = {}^t v_i \lambda_j v_j = \lambda_i {}^t v_i v_j \Rightarrow$

$(\lambda_i - \lambda_j) {}^t v_i v_j = 0 \Rightarrow {}^t v_i v_j = 0$.

④ $v \in \mathbb{C}^n \Rightarrow v = v_1 + \cdots + v_r$. $v_i \in V_{\lambda_i}$

$\Rightarrow {}^t v v = {}^t v_1 v_1 + \cdots + {}^t v_r v_r$

Suppose, for all $v_1 \in V_{\lambda_1}$, ${}^t v_1 v_1 = 0$. $\Rightarrow$ Consider $v_1^{\perp} = \{ v \in \mathbb{C}^n \mid {}^t v_1 v = 0 \}$. $\Rightarrow v_1^{\perp} = \mathbb{C}^n$, for $v_1^{\perp} \supset V_{\lambda_1}$, since, for $\forall w \in V_{\lambda_1}$ ${}^t(v_1 + w)(v_1 + w) = {}^t v_1 v_1 + 2 {}^t v_1 w + {}^t w w = 2 {}^t v_1 w = 0 \Rightarrow {}^t v_1 w = 0$

$\Rightarrow w \in v_1^{\perp}$, and $v_1^{\perp} \supset V_{\lambda_i}$, $i = 2, \cdots n$.

$\Rightarrow$ Contradiction, for we can always find $v \in \mathbb{C}^n$ s.t ${}^t v_1 v \neq 0$.

⑤ Thus we have $v_{11} \in V_{\lambda_1}$ s.t ${}^t v_{11} v_{11} \neq 0$.

In V_{λ_1}, $V_{\lambda_1} = \langle v_{11} \rangle \oplus v_1^{\perp}$. Continue $\cdots \Rightarrow$

$V_{\lambda_1} = \langle v_1 \rangle \oplus \cdots \oplus \langle v_{1j} \rangle$. $\Rightarrow {}^t v_{1i} v_{1j} = 0$, ${}^t v_{1k} v_{1k} \neq 0$