Thus, to prove the converse to Abel's theorem, we have to show that for $D = \Sigma(p_\lambda - q_\lambda)$ with $\mu(D)=0$, there exists a differential of the third kind $\eta$, holomorphic on $S - \{p_\lambda, q_\lambda\}$ with residues $a_\lambda$ at $p_\lambda$, $b_\lambda$ at $q_\lambda$, and having all integral periods. First, we check that we can at least find a meromorphic differential with the requisite singularities:

Lemma. Given a finite set of points $\{p_\lambda\}$ on $S$ and complex numbers $a_\lambda$ such that $\Sigma a_\lambda = 0$, there exists a differential of the third kind on $S$, holomorphic on $S - \{p_\lambda\}$ and having residue $a_\lambda$ at $p_\lambda$.

Proof. Consider the exact sheaf sequence on $S$

$$0 \longrightarrow \Omega^1 \longrightarrow \Omega^1(\Sigma p_\lambda) \xrightarrow{\text{res.}} \oplus\, \mathbb{C}_{p_\lambda} \longrightarrow 0.$$

$\ulcorner$ By p139, $\quad 0 \longrightarrow \mathcal{O}(E \otimes [-D]) \longrightarrow \mathcal{O}(E) \longrightarrow \mathcal{O}_D(E|_D) \rightarrow 0$.
Let $E = T^{*\prime}_S \otimes [D]$, where $D = \Sigma p_\lambda$.

$$0 \longrightarrow \underset{\Omega^1(S)}{\underset{\|}{\mathcal{O}(T^{*\prime}_S \otimes [D] \otimes [-D])}} \longrightarrow \underset{\Omega^1(\Sigma p_\lambda)}{\underset{\|}{\mathcal{O}(T^{*\prime}_S \otimes [D])}} \xrightarrow{\text{restriction}} \underset{\oplus\, \mathbb{C}_{p_\lambda}}{\underset{\|}{\mathcal{O}_D(E|_D)}} \rightarrow 0$$
$\lrcorner$

By Kodaira – Serre duality

$$H^1(S, \Omega^1) \cong H^0(S, \mathcal{O}) \cong \mathbb{C},$$

so that the image of $H^0(S, \Omega^1([\Sigma p_\lambda]))$ in $\oplus\, \mathbb{C}_{p_\lambda}$ has codimension at most 1.

$\ulcorner$. $H^1(S, \Omega^1(S \times \mathbb{C})) \cong H^0(S, \Omega^0(S \times \mathbb{C}^*)) \cong H^0(S, \mathcal{O}) \cong \mathbb{C}$.



$H^{0,0}(S) = H^0(S)$