$$\delta h \in H^2(\underline{U}, Z)$$
$$\delta h' \in H^2(\underline{U}, Z)$$

In $C^1(\underline{U}, \mathcal{O}^*)$. $\quad g \cdot g' = \{(g_{\alpha\beta}\, g'_{\alpha\beta},\ U_\alpha \cap U_\beta)\}$.

$$h + h' = \{(h_{\alpha\beta} + h'_{\alpha\beta},\ U_\alpha \cap U_\beta)\} \longrightarrow g \cdot g'$$

And. $\quad g \cdot g' \stackrel{\delta}{\longmapsto} \delta(h+h') = \delta h + \delta h'$.

$$L^* \otimes L = M \times \mathbb{C} \quad \text{trivial bundle}$$

$\Rightarrow \delta(L^* \otimes L) = 0$ since $g_{\alpha\beta} = 1$ for all α, β

$\Rightarrow h_{\alpha\beta} = 0$. or $2\pi n$. (constant). $\Rightarrow \delta h_{\alpha\beta} = 0$. $\Rightarrow \delta h = 0$.

$$\delta(L^* \otimes L) = \delta(L^*) + \delta(L) = c_1(L^*) + c_1(L) = 0$$

$\Rightarrow c_1(L^*) = -c_1(L)$. ⌟

Also, if $f : M \longrightarrow N$ is a holomorphic map of complex manifolds, the diagram

$$\begin{tikzcd}
H^1(M, \mathcal{O}^*) \arrow[r] & H^2(M, \mathbb{Z}) \\
H^1(N, \mathcal{O}^*) \arrow[r] \arrow[u, "f^*"] & H^2(N, \mathbb{Z}) \arrow[u, "f^*"']
\end{tikzcd}$$

Commutes, so that for $L \longrightarrow N$ any line bundle,

$$c_1(f^*L) = f^* c_1(L).$$

Γ

$$\begin{tikzcd}
C^1(\underline{U}, \mathcal{O}^*) \arrow[r, "f^*"] & C^1(f^{-1}(\underline{U}), \mathcal{O}^*) \\
g \arrow[r, mapsto] \arrow[d, "\delta"'] & f^*g \arrow[d, "\delta"] \\
C^2(\underline{U}, Z) \arrow[r, "f^*"] & C^2(f^{-1}(\underline{U}), Z) \\
\delta h \arrow[r, mapsto] & f^*\delta h = \delta(f^*g)
\end{tikzcd}$$