$$\frac{\partial f_1}{\partial t} = a x e^{at} + g_1(t, x, y) + t\frac{\partial g_1}{\partial t}$$
$$= a f_1(t, x, y) + [\partial]$$
$$= a x e^{at} + at\, g_1(t, x, y) + [\partial].$$

$$\Rightarrow \quad g_1(t, x, y) + t\frac{\partial g_1}{\partial t} = at\, g_1(t, x, y) + [\partial]$$

LHS has a nonzero x-term, but RHS does not, remember $a \neq 0$. In the same argument, $\frac{\partial g_1}{\partial y} = 0$ at $(x, y) = (0, 0)$. Thus $f_1(t, x, y) = x e^{at} + [\partial]$, where $[\partial]$ means terms involving x^2 or y^2. Similarly, we get $f_2(t, x, y) = y e^{at} + [\partial]$. Since $[\partial]$ contains x^2 or y^2 involving terms, we have only to consider $x e^{at}$ & $y e^{at}$ when we compute the Jacobian at the origin. $\Rightarrow$ $J_{f_t}(0) = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix}$.

$$f(t, z) = z e^{at} + [\partial]$$
$$J_{f_t}(0) = e^{at}$$

In general,

$$v(z_1, z_2) = a_{11} z_1 \frac{\partial}{\partial z_1} + a_{12} z_1 \frac{\partial}{\partial z_2} + a_{21} z_2 \frac{\partial}{\partial z_1} + a_{22} z_2 \frac{\partial}{\partial z_2} + [\partial]$$
$$= a_{11} x_1 \frac{\partial}{\partial x_1} + a_{11} y_1 \frac{\partial}{\partial y_1} + a_{12} x_1 \frac{\partial}{\partial x_2} + a_{12} y_1 \frac{\partial}{\partial y_2} + a_{21} x_2 \frac{\partial}{\partial x_1} + a_{21} y_2 \frac{\partial}{\partial y_1} + a_{22} x_2 \frac{\partial}{\partial x_2} + a_{22} y_2 \frac{\partial}{\partial y_2}$$

$$\frac{\partial f}{\partial t} = {}^t\Big(\overset{x_1}{\frac{\partial f_1}{\partial t}}, \overset{y_1}{\frac{\partial f_2}{\partial t}}, \overset{x_2}{\frac{\partial f_3}{\partial t}}, \overset{y_2}{\frac{\partial f_4}{\partial t}}\Big) = \begin{pmatrix} a_{11}, & 0, & a_{21}, & 0 \\ 0, & a_{11}, & 0, & a_{21} \\ a_{12}, & 0, & a_{22}, & 0 \\ 0, & a_{12}, & 0, & a_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} + [\partial]$$