⌈ $\Theta \in H^0(M, Z_d^2)$ since $\Theta$ is closed 2-form. ⌋

We see from the definition of $\delta_1$ that

$$\delta_1(\Theta) = \{\theta_\beta - \theta_\alpha\} \in Z^1(Z_d^1).$$

⌈ $\Theta \in C^0(\underline{U}, Z_d^2)$

$$\begin{tikzcd} 0 \arrow[r] & C^0(\underline{U}, Z_d^1) \arrow[r] \arrow[d] & C^0(\underline{U}, \mathcal{A}^1) \arrow[r, "d"] \arrow[d, "\delta"'] & C^0(\underline{U}, Z_d^2) \arrow[r] \arrow[d, "\delta"] & 0 \\ 0 \arrow[r] & C^1(\underline{U}, Z_d^1) \arrow[r] & C^1(\underline{U}, \mathcal{A}^1) \arrow[r] & C^1(\underline{U}, Z_d^2) \arrow[r] & 0 \end{tikzcd}$$

$$\theta \longrightarrow \Theta$$

$$\delta\theta \longrightarrow \delta\theta \longrightarrow 0$$

$$(\delta\theta)_{\alpha\beta} = -(\theta_\alpha - \theta_\beta) = \theta_\beta - \theta_\alpha.$$

$$\Rightarrow \quad \delta_1(\Theta) = (\delta\theta) = \delta\theta. \qquad ⌋$$

Now $\theta_\beta - \theta_\alpha = -d(\log g_{\alpha\beta})$, so

$$\delta_2\delta_1(\Theta) = \delta_2(\delta\theta) = \delta_2(\{\theta_\beta - \theta_\alpha\})$$

$$= \delta_2(\{-d(\log g_{\alpha\beta})\})$$

$$= \{-(\log g_{\alpha\beta} + \log g_{\beta\gamma} - \log g_{\alpha\gamma})\} = -2\pi i\, c_1(L).$$

⌈ $H^1(M, Z_d^1) \xrightarrow{\delta_2} H^2(\mathbb{R})$

$$\begin{tikzcd} 0 \arrow[r] & C^1(\underline{U}, \mathbb{R}) \arrow[r] & C^1(\underline{U}, \mathcal{A}^0) \arrow[r] \arrow[d] & C^1(\underline{U}, Z_d^1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^2(\underline{U}, \mathbb{R}) \arrow[r] & C^2(\underline{U}, \mathcal{A}^0) \arrow[r] & C^2(\underline{U}, Z_d^1) \arrow[r] & 0 \end{tikzcd}$$

$$-2\pi i\, h_{\alpha\beta} \qquad (\delta\theta)_{\alpha\beta} = -d(2\pi i\, h_{\alpha\beta})$$

$$0$$

$$\delta(-2\pi i\, h) \longmapsto \delta(-2\pi i\, h)$$

$$\Rightarrow \quad \delta_2(\delta\theta) = -2\pi i\, \delta h$$ see p287

$$\Rightarrow \quad -2\pi i\, c_1(L) = \delta_2(\delta_1(\Theta)) \qquad ⌋.$$