$$\Rightarrow \quad H^{n-m}(M) = \oplus L^k(\ker\Lambda \cap V_{2k+m})$$

since $\quad H^{n-m}(M) = V_m. \quad V_{2k+m} = H^{n-2k-m}(M).$

$$\Rightarrow \quad H^{n-m}(M) = \oplus L^k(\ker\Lambda \cap H^{n-2k-m}(M))$$

Let $\quad m = n-m.$

$$\Rightarrow \quad H^m(M) = \oplus L^k(\ker\Lambda \cap H^{m-2k}(M)) = \oplus L^k P^{m-2k}(M)$$ ⌟

Note that the Lefschetz decomposition is compatible with the Hodge decomposition; i.e., if we set

$$P^{p,q}(M) = (\ker\Lambda) \cap H^{p,q}(M),$$

then

$$P^\ell(M) = \bigoplus_{p+q=\ell} P^{p,q}(M).$$

Γ $\quad v \in P^\ell(M), \quad \Rightarrow \quad \Lambda v = 0$ and $\quad v \in H^\ell(M)$

$$\Rightarrow \quad v = \bigoplus_{p+q=\ell} v_{p,q}. \quad v_{p,q} \in H^{p,q}(M) = \mathcal{H}_d^{p,q}(M)$$

$$\Rightarrow \quad \Lambda v = \bigoplus_{p+q=\ell} \Lambda v_{p,q} = 0 \quad \Lambda v_{p,q} \in A^{p-1,q-1} \quad \Rightarrow \quad \Lambda v_{p,q} = 0$$ ⌟

We can give the following geometric interpretation of the Lefschetz theory in case the manifold M is embedded in projective space $\mathbb{P}^N$ with the induced metric. We have seen that the form (See Proof 2) P110

$$\omega = \frac{i}{2\pi} \partial\bar\partial \log \|Z\|^2$$ is closed and not exact

in $\mathbb{P}^N$. Since $H^2(\mathbb{P}^N)$ is one-dimensional, it follows