$$\eta_\omega \in H_{\bar\partial}^{n,n-1}(U^*) \cong H^{n-1}(U^*, \Omega^n),$$

and we have the

Residue Theorem

$$\sum_P \mathrm{Res}_p\, \omega = \int_{\partial M} \eta_\omega .$$

In particular, if M is compact, then

$$\sum_P \mathrm{Res}_p\, \omega = 0 .$$

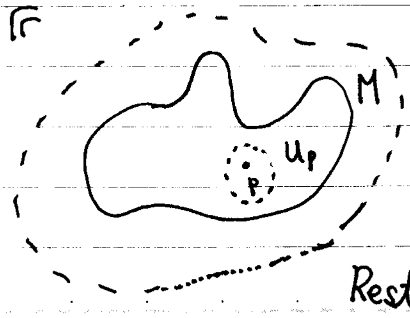
$U \supset \bar M$, $\bar U$ compact since M is relatively compact. $\partial M \subset U$.

Proof. As in the Riemann surface case, we let $U_p(\varepsilon)$ be an ε-ball around p and use $d\eta_\omega = 0$, and Stokes' theorem to write

$$\int_{\partial M} \eta_\omega = \sum_P \int_{\partial U_p(\varepsilon)} \eta\, \omega$$

$$= \sum_P \mathrm{Res}_p\, \omega \quad \text{by the above lemma,}$$

since $\eta_\omega | U_p^*$ is a Dolbeault representative of $[\omega | U_p] \in H^{n-1}(U_p^*, \Omega^n)$. Q.E.D.



$$0 = \int_{M - \cup U_{p}(\varepsilon)} d\eta_\omega \overset{\text{Stokes' theorem}}{=} \int_{\partial M} \eta_\omega - \sum_P \int_{\partial U_p(\varepsilon)} \eta_\omega$$

$$\Rightarrow \int_{\partial M} \eta_\omega = \sum_P \int_{\partial U_p(\varepsilon)} \eta_\omega$$

Restriction maps commute with sheaf maps.