$$\Rightarrow \quad \mathrm{Ext}^1(S; I, \mathcal{L}) \cong H^0(S, \underline{\mathrm{Ext}}^1_{\mathcal{O}}(I, \mathcal{L})) \oplus H^1(S, \mathcal{L})$$
$$= H^0(S, \underline{\mathrm{Ext}}^2_{\mathcal{O}}(\mathcal{O}_Z, \mathcal{L})) \oplus H^1(S, \mathcal{L})$$
$$\cong H^0(S, \mathcal{O}_Z) \oplus H^1(S, \mathcal{L})$$
$$\cong \bigoplus_{p \in Z} \mathcal{O}_{Z,p} \oplus H^1(S, \mathcal{L}).$$

$\Rightarrow$ We may find $e \in \mathrm{Ext}^1(S; I, \mathcal{L})$ s.t e_p is a unit for each $p \in Z$. $\Rightarrow$ (**) may be solved.
Remember that the isomorphisms are induced by the natural maps such as inclusions, restrictions, etc. ⌟

Thus: If $H^2(S, \mathcal{L}) = 0$, then we may find a rank-two holomorphic vector bundle $E \longrightarrow S$ and section $s \in H^0(S, \mathcal{E})$ such that $c_1(\mathcal{L}) = -c_1(\mathcal{E})$ and s defines Z ideal-theoretically. In particular, we may take $\mathcal{L} = \mathcal{O}$ in case $p_g(S) = 0$. Taking $\mathcal{L}$ to be sufficiently ample, we may always arrange that $H^2(S, \mathcal{L}) = 0$, so that our original problem will have at least one solution.

⌜ $H^2(S, \mathcal{L}) = 0 \Rightarrow$ We may find $e \in \mathrm{Ext}^1(S; I, \mathcal{L})$ s.t. e_p is a unit in $\underline{\mathrm{Ext}}^1(I, \mathcal{L})_p$ for each $p \in Z$.
$\Rightarrow \exists$ $\mathcal{E}^*$ locally free s.t

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E}^* \longrightarrow I \longrightarrow 0$$

$\Rightarrow \mathcal{E}^* = \mathcal{O}(E^*)$, for a rank-two holomorphic vector bundle $E \longrightarrow S$ with $c_1(\mathcal{L}) = -c_1(E) = c_1(E^*)$ by P726 ~ P727.