

$$\sum_{\mathbf{z}} |\mathbf{z}|^{2s+4} |u_{\mathbf{z}}|^2 < \infty \iff \sum_{\mathbf{z}} (1 + |\mathbf{z}|^2)^{s+2} |u_{\mathbf{z}}|^2 < \infty$$

Since $1 + |\mathbf{z}|^2 < 2|\mathbf{z}|^2$ and $|\mathbf{z}|^2 + 1 > |\mathbf{z}|^2$.

Set $v = \sum_{\mathbf{z} \neq 0} i\mathbf{z} u_{\mathbf{z}} e^{i\mathbf{z}x}$.

$$\Rightarrow \|v\|_{s+1}^2 = \sum_{\mathbf{z}} (1 + |\mathbf{z}|^2)^{s+1} |v_{\mathbf{z}}|^2 = \sum_{\mathbf{z}} (1 + |\mathbf{z}|^2)^{s+1} |\mathbf{z}|^2 |u_{\mathbf{z}}|^2$$

$$\leq \sum_{\mathbf{z}} (2|\mathbf{z}|^2)^{s+1} |\mathbf{z}|^2 |u_{\mathbf{z}}|^2 = 2^{s+1} \sum_{\mathbf{z}} |\mathbf{z}|^{2s+4} |u_{\mathbf{z}}|^2 < \infty.$$

$\Rightarrow v \in H_{s+1}$, and therefore, $v \in C^s(T)$ by induction hypothesis.

The convergence being uniform, we may integrate term by term:

$$\begin{aligned} \int_0^x v(t) dt &= \int_0^x \sum_{\mathbf{z}} i\mathbf{z} u_{\mathbf{z}} e^{i\mathbf{z}t} dt \\ &= \sum_{\mathbf{z}} i\mathbf{z} u_{\mathbf{z}} \int_0^x e^{i\mathbf{z}t} dt = \sum_{\mathbf{z}} i\mathbf{z} u_{\mathbf{z}} \left[\frac{e^{i\mathbf{z}t}}{i\mathbf{z}} \right]_0^x = \sum_{\mathbf{z}} u_{\mathbf{z}} (e^{i\mathbf{z}x} - 1) \\ &= \sum_{\mathbf{z} \neq 0} u_{\mathbf{z}} e^{i\mathbf{z}x} - \sum_{\mathbf{z} \neq 0} u_{\mathbf{z}} = \cancel{u(x)} - u_0 - (u(0) - u_0) \\ &= u(x) - u(0). \end{aligned}$$

$$\Rightarrow u'(x) = v(x) \text{ and } u \in C^{s+1}(T).$$

For $n=2$ case.

$$u = \sum_{\mathbf{z} \in \mathbb{Z}^2} u_{\mathbf{z}} e^{i\langle \mathbf{z}, x \rangle}$$

$$v_1 = \sum_{\mathbf{z}_1 \neq 0} i\mathbf{z}_1 u_{\mathbf{z}_1} e^{i\langle \mathbf{z}_1, x \rangle}, \quad v_2 = \sum_{\mathbf{z}_2 \neq 0} i\mathbf{z}_2 u_{\mathbf{z}_2} e^{i\langle \mathbf{z}_2, x \rangle}$$

$$\Rightarrow \frac{\partial u}{\partial x_i} = \sum_{\mathbf{z} \in \mathbb{Z}^2} u_{\mathbf{z}} (i\mathbf{z}_i) e^{i\langle \mathbf{z}, x \rangle}$$

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Q.E.D.