Proof. By the hard Lefschetz and Kodaira – Serre duality, the pairing

$$H^{1,0}(M) \otimes H^{0,1}(M) \longrightarrow \mathbb{C}$$

given by

$$\varphi \otimes \psi \longrightarrow \int_M \omega^{n-1} \wedge \varphi \wedge \psi$$

is nondegenerate.

∏

$$\begin{array}{ccc} H^{1,0}(M) & \xrightarrow[\cong]{L^{n-1}} & H^{n,n-1}(M) \\ \cap & & \cap \\ H^1(M) = H^{n-(n-1)} & \xrightarrow{\cong} & H^{2n-1}(M) = H^{n+(n-1)}(M) \\ \varphi & \longmapsto & \omega^{n-1} \wedge \varphi \end{array}$$

See also p 123.

since $L^{n-1}: H^{n-(n-1)} \longrightarrow H^{2n-1}(M)$ is isomorphic by Lefshetz hard theorem and L^{n-1} is compatible with Hodge decomposition ($\Rightarrow$ $H^{p,q} \cong H^{p+k,\, q+k}$, where $p+q = n-k$)

And, then

$$H^{0,1}(M) \otimes H^{n,n-1}(M) \cong H^1(M, \mathcal{O}) \otimes H^{n-1}(M, \Omega^n) \longrightarrow \mathbb{C}$$

is nondegenerate by Kodaira – Serre duality. P 102. ⅃

According to the formal rules for $\iota(v)$ listed above, for all $\varphi \in H^{1,0}(M)$,

$$0 = \iota(v)(\omega^n \wedge \varphi) \qquad (\text{since } \omega^n \wedge \varphi \equiv 0 \text{ for trivial reasons})$$

$$= n\, \omega^{n-1} \wedge \iota(v)\omega \wedge \varphi$$

$$\Rightarrow \int_M \omega^{n-1} \wedge \iota(v)\omega \wedge \varphi = 0$$

$$\Rightarrow \iota(v)\omega = 0 \text{ in } H^{0,1}(M)$$

by the non degeneracy of the pairing. Q.E.D. for lemma.