assume that V is nondegenerate in $\mathbb{P}^k$, $k<n$.
Apply the argument above.

"

⌟

By the argument above, $V_C \sim I_{l_1} + I_{l_2} + I_{p_1} + I_{p_2}$.
If we see how V_C degenerates when C degenerates into $l_1 + l_2$, we may see the two conditions are equivalent, more convincingly.

This is the reason why the two conditions are equivalent, "some sort of".

⌟

(In fact, the blow-up W may be constructed geometrically as follows: let W^* denote the linear system of conics in $\mathbb{P}^{2*}$, $W_1^* \subset W^*$ the locus of singular conics, and take the closure in $W \times W^*$ of the locus

$$\{(C, D) : D = C^*\} \subset (W - W_1) \times (W^* - W_1^*).$$

A pair (C, D) in this closure was classically called a complete conic.).

⌜ Actually, since the blow-up counts the tangent of a curve, and a dual curve is defined in terms of tangent lines of a curve, see P264, the construction above is valid. ⌟