degree n passing through Γ_0, see P138.

$\Rightarrow$ $$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}([nH - C_0]) \xrightarrow{\otimes S_0} \mathcal{O}_{\mathbb{P}^2}(nH) \xrightarrow{\text{res}} \mathcal{O}_{\Gamma_0}(nH) \longrightarrow 0$$
↗ restriction

is exact, for given $\forall\ \sigma \in \mathcal{O}_{\mathbb{P}^2}(nH)$ s.t $\sigma = 0$ on Γ_0, σ must be a curve of degree n passing through Γ_0, and so $\sigma \in \mathcal{O}_{\mathbb{P}^2}([nH - C_0])$, where $(S_0 = 0) = C_0$. Here the exact sequence holds for a singular curve C_0.

$\Rightarrow$ $$0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(nH - C_0)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(nH)) \longrightarrow H^0(\Gamma_0, \mathcal{O}(nH))$$
$$\longrightarrow H^1(\mathbb{P}^2, \mathcal{O}(nH - C_0)) \longrightarrow \underbrace{H^1(\mathbb{P}^2, \mathcal{O}(nH))}_{= 0 \text{ by P156.}}$$

$\Rightarrow$ $$\dim H^0(\mathbb{P}^2, \mathcal{O}(nH - C_0)) - \dim H^0(\mathbb{P}^2, \mathcal{O}(nH)) + \dim H^0(\Gamma_0, \mathcal{O}(nH))$$
$$- \dim H^1(\mathbb{P}^2, \mathcal{O}(nH - C_0)) = 0$$

$\Rightarrow$ Since $\dim H^0(\mathbb{P}^2, \mathcal{O}(nH - C_0)) = \dim |\mathcal{J}_{\Gamma_0}(nH)| + 1$ P137

$\dim H^0(\mathbb{P}^2, \mathcal{O}(nH)) = {}_{n+2}C_2 = \frac{(n+2)(n+1)}{2}$, $h^0(\Gamma_0, \mathcal{O}(nH)) = \deg \Gamma_0 = d$

$\dim H^1(\mathbb{P}^2, \mathcal{O}(nH - C_0)) = h^1(\mathcal{J}_{\Gamma_0}(n))$ (some sort of notation).

we get
$$\dim |\mathcal{J}_{\Gamma_0}(n)| + 1 = \frac{(n+2)(n+1)}{2} - d + h^1(\mathcal{J}_{\Gamma_0}(n))$$

$\Leftrightarrow$
$$\dim |\mathcal{J}_{\Gamma_0}(n)| = \frac{n(n+3)}{2} - d + h^1(\mathcal{J}_{\Gamma_0}(n))$$
$$= \frac{n(n+3)}{2} - d + \omega$$

will be explained more later

$\Rightarrow$ $\omega = h^1(\mathcal{J}_{\Gamma_0}(n)) = \dim H^1(\mathbb{P}^2, \mathcal{O}(nH - C_0)) = h^1(\mathcal{O}_{\mathbb{P}^2}(nH - C_0))$

By Riemann-Roch
$$\chi(\mathcal{O}_S(L)) = \frac{1}{2}(L \cdot L - K_S \cdot L) + \chi(\mathcal{O}_S)$$
$$= \frac{n(n+3)}{2} - d + \chi(\mathcal{O}_S)$$