Look at

$$\int_{\mathbb{R}^n} \phi\psi \wedge (P\eta)\, dx_I = \int_{\mathbb{R}^n} \phi\varphi \wedge \eta\, dx_I.$$

Remember ψ is $(n - \#I = n-p)$-form. $\Rightarrow$ Forget dx_I's.

$\Rightarrow$

$$(*) \cdots \int_{\mathbb{R}^n} \phi\psi \ P(\eta) = \int_{\mathbb{R}^n} \phi\varphi\,\eta \quad \text{for all } \eta \in C^\infty(\mathbb{R}^n)$$

$\phi\psi$ & $\phi\varphi$ have the compact support.

Thus if we prove the following:

"If $\int_{\mathbb{R}^n} \psi \ P(\eta)\, dx = \int_{\mathbb{R}^n} \varphi\,\eta\, dx \quad$ for all $\eta \in C_c^\infty(\mathbb{R}^n)$

, where ψ, φ are compactly supported, then, in case $\psi \in \mathcal{H}_0$ & $\varphi \in \mathcal{H}_s$, we can deduce that $\psi \in \mathcal{H}_{s+2}$."

then it is clear that if $(*)$ is true, then

$\phi\psi \in \mathcal{H}_{s+2} \Rightarrow \psi \in \mathcal{H}_{s+2}(loc) \Rightarrow \psi \in \mathcal{H}_{s+2}(M)$

One more time; if we have ψ, φ s.t

$\langle \psi, \Delta\eta \rangle = \langle \varphi, \eta \rangle$ for all η.

$\Rightarrow$ Locally, say open set $U_\alpha \subset M$, $\overline{V_\alpha} \subset U_\alpha$.

$\Rightarrow \exists \{\phi_\alpha\}$ s.t $\operatorname{supp}\phi_\alpha \subset U_\alpha$ and $\phi_\alpha \neq 0$ on $\overline{V_\alpha}$

$\sum \phi_\alpha = 1$, & $\exists \ \psi_\alpha = 1$. $\operatorname{supp}\psi_\alpha \subset U_\alpha$

$$\Rightarrow \int_{V_\alpha} \phi_\alpha \psi \wedge \Delta \psi_\alpha \eta = \int_{V_\alpha} \phi_\alpha \varphi \wedge \psi_\alpha \eta$$