Inequalities of the type

$2\alpha\beta \leq \epsilon\alpha^2 + \frac{1}{\epsilon}\beta^2$ will be used repeatedly,

and $\Phi = C_n \Phi' \wedge \bar{\Phi}$ denote the volume form. Set

$$\eta = C_n'((\bar{\nabla}\psi, \psi) \wedge \omega^{n-1})$$
$$= C_n\Big(-\sum_{\substack{I,J,K\\ i}} (-1)^{K-1} \psi_{I\bar{J},i,\bar{K}} \overline{\psi_{I\bar{J},j}}\, g_{i\bar{j}}\, \varphi_1 \wedge \cdots \wedge \varphi_K \wedge \cdots \wedge \varphi_n\Big) \wedge \Phi'$$
$$+ \text{ lower order terms.}$$

where $g_{i\bar{j}} = \langle e_i, e_j\rangle$, e_i's holomorphic frame for E.

$$\Rightarrow \partial\eta = \Big(-2\sum_{\substack{I,J,K\\ i,j}} \psi_{I\bar{J},i\bar{K},K} \overline{\psi_{I\bar{J},j}}\, g_{i\bar{j}}\Big)\Phi$$
$$-\Big(2\sum_{\substack{I,J,K\\ i,j}} \psi_{I\bar{J},i,\bar{K}} \overline{\psi_{I\bar{J},j,\bar{K}}}\, g_{i\bar{j}}\Big)\Phi$$
$$-\sum_{\substack{I,J,K\\ i,j}} (-1)^{K-1} \psi_{I\bar{J},i\bar{K}} \overline{\psi_{I,\bar{J},j}}\, (g_{i\bar{j}})_K\, \varphi_K \wedge \varphi_1 \wedge \cdots \wedge \hat{\varphi}_K \wedge \cdots \wedge \varphi_n)\wedge\Phi'$$
$$+ \text{ lower order terms}$$

Thus, by the Weitzenböck formula

$$(\Delta\psi)_{I\bar{J},i} = -2\sum_{K=1}^{n} \nabla_K \nabla_{\bar{K}} \psi_{I\bar{J}.i} + (A'(\psi))_{I,\bar{J},i},$$

$$\partial\eta = \Big(\sum (\Delta\psi)_{I\bar{J}.i} \overline{\psi_{I.\bar{J}.j}}\, g_{i\bar{j}}\Big)\Phi$$
$$-\Big(\sum \psi_{I\bar{J}\bar{K}} \overline{\psi_{I\bar{J}\bar{K}}}\, g_{i\bar{j}}\Big)\Phi + (A'\psi.\psi)\Phi.$$

$$\Rightarrow \langle\Delta\psi.\psi\rangle = \|\bar{\nabla}\psi\|^2 + \langle A'\psi.\psi\rangle.$$