$\ulcorner$ $DT_\epsilon(\varphi) = (-1)\, T_\epsilon(D\varphi) = T_\epsilon(-D\varphi)$ by P367 Example 2.

$$= \int -\frac{\partial \varphi}{\partial x_i}(x)\; T_\epsilon(x)\, dx \underset{\Uparrow}{=} \iint -\frac{\partial \varphi}{\partial x_i}(x)\; \psi(y)\, \chi_\epsilon(x-y)\, dy\, dx$$

(we assumed $T = T_\psi$)

$$= \iint -\frac{\partial \varphi}{\partial x_i}(x)\; \psi(x-u)\, \chi_\epsilon(u)\, du\, dx, \quad (\text{let } u = x-y.)$$

$$= \iint -\frac{\partial \varphi}{\partial x_i}(x)\; \psi(x-u)\, \chi_\epsilon(u)\, dx\, du, \quad (\text{by Fubini.})$$

$$= \int \chi_\epsilon(u) \left(\int -\frac{\partial \varphi}{\partial x_i}(x)\; \psi(x-u)\, dx \right) du$$

$$= \int \chi_\epsilon(u) \left(\int \varphi(x)\; \frac{\partial \psi(x-u)}{\partial x_i}\, dx \right) du \quad (\text{by integration by parts})$$

$$= \iint \chi_\epsilon(u)\, \varphi(x)\; \frac{\partial \psi}{\partial x_i}(x-u)\, dx\, du$$

$$= \iint \chi_\epsilon(u)\, \varphi(x)\; \frac{\partial \psi}{\partial x_i}(x-u)\, du\, dx. \quad (\text{by Fubini})$$

$$= \iint \chi_\epsilon(x-y)\, \varphi(x)\; \frac{\partial \psi}{\partial y_i}(y)\, dy\, dx. \quad (\text{let } y = x-u)$$

$$= \iint \chi_\epsilon(x-y)\, \varphi(x)\; \frac{\partial \psi}{\partial y_i}(y)\, dx\, dy \quad (\text{by Fubini})$$

$$= \int \left(\int \varphi(x)\; \chi_\epsilon(x-y)\, dx \right) \frac{\partial \psi}{\partial y_i}(y)\, dy \quad \cdots \quad (*)$$

$$\Rightarrow \quad (DT)_\epsilon(\varphi) = \int \varphi(x)\; (DT)_\epsilon(x)\, dx = \int \varphi(x)\; (T_{D\psi})_\epsilon(x)\, dx$$

$$= \int \varphi(x) \int D\psi(y)\, \chi_\epsilon(x-y)\, dy\, dx$$

$$= \int \varphi(x)\; \chi_\epsilon(x-y)\; \frac{\partial \psi}{\partial y_i}(y)\, dy\, dx \quad \cdots \quad (**)$$

$$\Rightarrow (*) = (**) \quad \Rightarrow \quad (DT_\epsilon)(\varphi) = (DT)_\epsilon(\varphi). \quad \lrcorner$$