$$\nabla\psi = \nabla'\psi + \nabla''\psi.$$

$$\|\nabla\psi\|^2 = \|\nabla'\psi\|^2 + \|\nabla''\psi\| \quad \text{since} \quad \langle \varphi_i, \overline{\varphi_j}\rangle = 0.$$

Remark.1. From $\langle \Delta\psi, \psi\rangle = \|\bar\nabla\psi\|^2 + \langle A'\psi, \psi\rangle$,

$$\langle \Delta\psi, \psi\rangle - \|\bar\nabla\psi\|^2 = \langle A'\psi, \psi\rangle$$

$$\Rightarrow 2|\langle \Delta\psi, \psi\rangle - \|\bar\nabla\psi\|^2| = 2|\langle A'\psi, \psi\rangle| \le 2\|A'(\psi)\|\,\|\psi\|$$

$$\le \epsilon\|A'(\psi)\|^2 + \frac{1}{\epsilon}\|\psi\|^2 \le \epsilon(K\|\bar\nabla\psi\|^2 + K\|\psi\|^2) + \frac{1}{\epsilon}\|\psi\|^2$$

$$\Rightarrow 2\langle \Delta\psi, \psi\rangle \le (2+\epsilon K)\|\bar\nabla\psi\|^2 + (\epsilon K + \tfrac{1}{\epsilon})\|\psi\|^2$$

$$\frac{1}{2}\cdot 2\langle \Delta\psi, \psi\rangle \le \frac{2+\epsilon K}{2}\|\bar\nabla\psi\|^2 + \frac{\epsilon K + \frac{1}{\epsilon}}{2}\|\psi\|^2$$

$$\Rightarrow \langle \Delta\psi, \psi\rangle \le M(\|\bar\nabla\psi\|^2 + \|\psi\|^2)$$

$$\le M(\|\nabla\psi\|^2 + \|\bar\nabla\psi\|^2 + \|\psi\|^2)$$

$$\Rightarrow \underbrace{\langle \Delta\psi, \psi\rangle + \langle \psi, \psi\rangle}_{=\mathcal{D}(\psi)} \le M'(\|\nabla\psi\|^2 + \|\bar\nabla\psi\|^2 + \|\psi\|^2)$$

$$\mathcal{D}(\psi) \le C\|\psi\|_1^2$$

This implies that the Dirichlet norm is equivalent to Sobolev 1-norm.

Remark.2. In the Kähler case, one may use the precise Weitzenböck formular, and the above integration by parts calculation to prove the Kodaira identity

$$\langle \Delta\psi, \psi\rangle = \|\bar\nabla\psi\|^2 + (R\psi, \psi)$$

where, for $\psi \in A^{0,q}(M)$ and summing repeated indices,

$$\langle R\psi, \psi\rangle = q\int_M (R_{i\bar j}\psi_{\bar i_1\cdots\bar i_{q-1}\bar i}\overline{\psi_{\bar i_1\cdots\bar i_{q-1}\bar j}})\,\Phi.$$