On the other hand, H is a polynomial of the form

$$h(z)^d + a_1(w)\, h(z)^{d-1} + \cdots + a_d(w) \qquad (w = f(z)),$$

whose coefficients are holomorphic functions of $w = f(z)$.
Now, via the mapping $f: U \longrightarrow W$ the local ring $\mathcal{O}_w = \{$ germ-s of holomorphic functions $h(w)$ defined in some nbd of $w=0\}$ injects into $\mathcal{O}_z$, and we have proved

The degree of the extension $[\mathcal{O}_z : \mathcal{O}_w] = d$; i.e., every $h \in \mathcal{O}_z$ satisfies a polynomial equation of degree $\leq d$ with coefficients in $f^*\mathcal{O}_w \subset \mathcal{O}_z$ and moreover d is the least such integer.

$\ulcorner$

$$\begin{array}{ccc} \mathcal{O}_w & \longrightarrow & \mathcal{O}_z \\ \cup & & \cup \\ g & \longmapsto & f^*g \end{array}$$

Suppose $f^*g \equiv 0$ on some nbd of 0.
$\Rightarrow$ Since f is open, $g \equiv 0$ on some nbd of 0. $\Rightarrow$ $g \equiv 0$ in $\mathcal{O}_w$ $\Rightarrow$ f^* is injective.
For each i, $a_i(w) = a_i(f(z)) = (f^*a_i)(z)$. $\Rightarrow f^*a_i \in f^*\mathcal{O}_w \subset \mathcal{O}_z$.

For example $n=2$. Consider $h: U \longrightarrow \mathbb{C}$ defined by $(z_1, z_2) \longmapsto z_2$. Suppose $d' < d$ and

$$z_2^{d'} + a_1(w)\, z_2^{d'-1} + \cdots + a_{d'}(w) = 0.$$