Warning! Definition of Q on $H^m(M) = \sum L^k P^{m-2k}(M)$.

$$\xi \in H^m(M) \Rightarrow \xi = \oplus L^k \xi_{m-2k}$$
$$\eta \in H^m(M) \Rightarrow \eta = \oplus L^k \eta_{m-2k}$$

$$\Rightarrow \quad Q(\xi,\eta) = \sum_k Q(L^k \xi_{m-2k}, L^k \eta_{m-2k})$$

$\Downarrow$ by P228. Key Point.

$$Q(\xi,\eta) = \sum_{k,l} Q(L^k \xi_{m-2k}, L^l \eta_{m-2l})$$ See P202. Wells.

If $\xi = \oplus L^k \xi_{m-2k} \in H^m(M)$ s.t $Q(\xi,\eta) = 0$ for all $\eta \in H^m(M)$, choose $\eta = L^k \bar{\xi}_{m-2k}$.

$$\Rightarrow \quad Q(\xi, L^k \bar{\xi}_{m-2k}) = Q(L^k \xi_{m-2k}, L^k \bar{\xi}_{m-2k})$$
$$= Q(\xi_{m-2k}, \bar{\xi}_{m-2k}) \neq 0 \quad \text{if } \xi_{m-2k} \neq 0$$
$$\Rightarrow \xi_{m-2k} = 0$$

Similarly, we can show $\xi_{m-2k} = 0$ for all k.

$\Rightarrow \xi = 0$.

This proves that Q is non-degenerate.

Comment: On P123, Griffiths said that he can define a bilinear form

$$Q : H^{n-k}(M) \otimes H^{n-k}(M) \longrightarrow \mathbb{C}$$

by setting $\quad Q(\xi,\eta) = \int_M \xi \wedge \eta \wedge \omega^k$.

* Key Point *

$$Q(L\xi,\eta) = \int L\xi \wedge \eta \wedge \omega^k, \quad \xi \in P^{n-k-2}(M) = V_{k+2}, \quad \eta \in P^{n-k},$$
$$= \int \xi \wedge \omega \wedge \eta \wedge \omega^k = \pm \int \xi \wedge \eta \wedge \omega^{k+1} = \pm \int \xi \wedge L^{k+1}\eta = 0$$

since $L^{k+1}\eta = 0 \quad (\because \eta \in P^{n-k})$