$\ulcorner$ $\Theta_\alpha = d\,\theta_\alpha - \theta_\alpha \wedge \theta_\alpha$ $\Rightarrow$ Since θ_α is 1-form,

$$d\,\Theta_\alpha = -d\theta_\alpha \wedge \theta_\alpha + \theta_\alpha \wedge d\theta_\alpha$$
$$= -d\theta_\alpha \wedge \theta_\alpha + d\theta_\alpha \wedge \theta_\alpha = 0. \quad \lrcorner$$

Recall also that for any analytic subvariety $V \subset M$ of dimension k, we have defined the fundamental class $(V) \in H_{2k}(M, \mathbb{R})$ to be given by the linear functional

$$\varphi \longmapsto \int_V \varphi \quad \text{on} \quad H^{2k}_{DR}(M);$$

we denote its Poincare dual by η_V. In particular, we take the fundamental class of a divisor $D = \Sigma a_i V_i$ on M to be $\Sigma a_i (V_i)$; we denote its Poincare dual by

$$\eta_D = \Sigma a_i \eta_{V_i}.$$

This subsection will be devoted to proving the

Proposition 1. For any line bundle L with curvature form Θ,

$$c_1(L) = \frac{i}{2\pi} \Theta \in H^2_{DR}(M).$$

2. If $L = [D]$ for some $D \in Div(M)$,

$$c_1(L) = \eta_D \in H^2_{DR}(M).$$

pf). First, we unwind the definition of $c_1(L)$ for $L \longrightarrow M$ a line bundle with trivializations φ_α and transition functions $g_{\alpha\beta}$ relative to a cover $\underline{U} = \{U_\alpha\}$ of M. We may assume the open sets U_α are simply connected and set

$$h_{\alpha\beta} = \frac{1}{2\pi i} \log g_{\alpha\beta}.$$