$\phi \circ \psi$ = identity & $\psi \circ \phi = id \Rightarrow \phi$ is isomorphic

$\Rightarrow \mathcal{O}^{(2)}/m\mathcal{O}^{(2)} \cong \mathcal{O}/m\mathcal{O} \oplus \mathcal{O}/m\mathcal{O} \cong \mathbb{C} \oplus \mathbb{C} \cong \mathbb{C}^2$, since $m\mathcal{O}$ is the maximal ideal m.

Consider the following map β_0:

$$\beta_0 : M/mM \longrightarrow \mathcal{O}^{(2)}/m\mathcal{O}^{(2)}$$

$$g_1 f_1 + g_2 f_2 + mM \longmapsto (g_1, g_2) + m\mathcal{O}^{(2)}.$$

Suppose $g_1 f_1 + g_2 f_2 + mM = g_1' f_1 + g_2' f_2 + mM$.

$\Rightarrow (g_1 - g_1') f_1 + (g_2 - g_2') f_2 \in mM$

$\Rightarrow$ If $g_1 - g_1' \notin m$, then $g_1 - g_1'$ is a unit. $\Rightarrow$ Since $(g_1 - g_1') f_1 + (g_2 - g_2') f_2 = m_1 f_1 + m_2 f_2$, $f_1 = -(g_1 - g_1')^{-1}(g_2 - g_2') f_2 + m_1' f_1 + m_2' f_2 \Rightarrow f_1$ may be expressed as a multiple of f_2.

More clearly, $(g_1 - g_1') f_1 + (g_2 - g_2') f_2 = m_1 f_1 + m_2 f_2$

$\Rightarrow (g_1 - g_1' - m_1) f_1 + (g_2 - g_2' - m_2) f_2 = 0$

$\Rightarrow$ Since $(g_1 - g_1')(0) \neq 0$, $(g_1 - g_1' - m_1)(0) \neq 0$.

$\Rightarrow f_1 = -(g_1 - g_1' - m_1)^{-1}(g_2 - g_2' - m_2) f_2 \Rightarrow$ Contradiction to the minimality of $k=2 \Rightarrow g_1 - g_1' \in m$ & $g_2 - g_2' \in m$.

$\Rightarrow \beta_0$ is well-defined.

$\alpha_0 \circ \beta_0 (g_1 f_1 + g_2 f_2 + mM) = \alpha_0((g_1, g_2) + m\mathcal{O}^{(2)}) = g_1 f_1 + g_2 f_2 + mM$

$\beta_0 \circ \alpha_0 ((g_1, g_2) + m\mathcal{O}^{(2)}) = (g_1, g_2) + m\mathcal{O}^{(2)}$.

$\Rightarrow \alpha_0$ is isomorphic

$$\begin{array}{ccc} \mathcal{O}^{(k)} & \xrightarrow{\alpha} & M = \{f_1 \cdots f_k\} \quad k \text{ minimal} \\ \Downarrow & & \Downarrow \\ e_i = (0, \cdots \overset{i}{1} \, 0 \cdots 0) & \longmapsto & f_i \end{array}$$