We shall not use distributions in proving the Hodge theorem, but they will be rather extensively discussed in §1 of Chapter 3. Note in passing that the equation

$\Delta_d \varphi = \psi$, where ψ is a distribution, may be solved provided that $\psi(\eta) = 0$ for any harmonic η. If $\psi \in H_s$ for any s, positive or negative, then $\varphi \in H_{s+2}$.

In particular, regularity holds for distribution solutions as well as weak Hilbert-space solutions. We shall work in this latter setting in order to take advantage of the standard theory of compact, self-adjoint operators on Hilbert spaces.

Proof of the Hodge Theorem II: Global Theory.

On a torus, the Sobolev s-norm is given equivalently by a weighted Fourier series norm or by the L^2-norm

$$\sum_{|\alpha| \le s} \int_T |D^\alpha \varphi|^2 dx .$$

$$\varphi = \sum \varphi_\xi e^{i\langle \xi, x\rangle}$$

$$D^\alpha \varphi = \sum (D^\alpha \varphi)_\xi e^{i\langle \xi, x\rangle} = \sum \xi^\alpha \varphi_\xi e^{i\langle \xi, x\rangle}$$

$$\|D^\alpha \varphi\|_0^2 = \sum |(D^\alpha\varphi)_\xi|^2 = \sum \int_T (D^\alpha\varphi)_\xi \overline{(D^\alpha\varphi)_\xi}\, dx$$

$$= \int_T D^\alpha\varphi\, \overline{D^\alpha\varphi}\, dx = \int_T \sum_\xi (D^\alpha\varphi)_\xi \sum \overline{(D^\alpha\varphi)_\xi}\, dx$$

This latter may be extended to vector bundles over manifolds so that the Sobolev lemma and Rellich lemma both remains valid. We now explain how this is done.