We use $|a-b|^2 \leq 2(|a|^2+|b|^2)$.
$\Rightarrow$ We proved that $\{u_k\}$ is a Cauchy sequence.
$\Rightarrow$ Since H_r is Hilbert space, $\{u_k\}$ converges in H_r.
Q.E.D.

Comment: We need not use this trick. Use my argument on P113.

We now wish to examine the ~~Laplacian~~ Laplace equation on the torus T. Essentially we are going to prove in this case the Hodge theorem for 0-forms, or functions. with the standard Euclidean metric and relative to the exterior derivative d.
Although it is probably unnecessary, we remark that on a compact Riemannian man. M, we may define the adjoint d^* of d, form the Laplacian $dd^* + d^*d = \Delta_d$, and arrive at the exact same formalism as for $\bar\partial$ on complex manifold. The Hodge theorem is, of course, also true, and the proof is the same as the one proof is the same as the one we shall give in the complex case.

For $\varphi \in C^\infty(T)$, the Laplacian is

$$\Delta_d \varphi = \sum_i D_i^2 \varphi = -\sum_i \frac{\partial^2}{\partial x_i^2}(\varphi) = \sum_i \sum \varphi_\xi \xi_i^2 e^{i\langle\xi,x\rangle} = \sum_\xi \varphi_\xi \|\xi\|^2 e^{i\langle\xi,x\rangle}$$

since $\varphi = \sum_\xi \varphi_\xi e^{i\langle\xi,x\rangle}$

$$\left(d\underbrace{d^*}_{=0} + d^*d = -\sum_i \frac{\partial^2}{\partial x_i^2} \quad \text{see p83} \right)$$

We will discuss the equation (*) $\Delta_d \varphi = \psi$ in a manner s.t the conclusions carry over to a general compact manifold. A function $\varphi \in L^2(T) = H_0$ is said to be a weak solution to (*) iff $\langle \Delta_d \eta, \varphi) = (\eta, \psi)$
$\varphi \overset{L^2\text{-sense}}{=} \sum \varphi_\xi e^{i\langle\xi,x\rangle}$