$$\chi(\mathcal{O}_{\tilde{S}}) = \underset{\substack{\| \\ 1}}{h^{0,0}(\tilde{S})} - \underset{\substack{\| \\ h^{1,0}(\tilde{S}) \\ \| \\ q=0}}{h^{0,1}(\tilde{S})} + \underset{\substack{\| \\ h^{2,0}(\tilde{S}) \\ \| \\ p_g=0}}{h^{0,2}(\tilde{S})} \quad \text{by P246}$$

The blow-up of Kähler is again Kähler by 192. Coro.

$$\Rightarrow \chi(\mathcal{O}_{\tilde{S}}(L)) = \frac{n^2+3n}{2} - d + 1.$$

$$\chi(\mathcal{O}_{\tilde{S}}(L)) = h^0(\tilde{S}, \mathcal{O}(L)) - \underset{\substack{\| \\ h^1(\mathcal{O}_{\tilde{S}}(L))}}{h^1(\tilde{S}, \mathcal{O}(L))} + \underset{\substack{\| \\ h^2(\mathcal{O}_{\tilde{S}}(L)) = h^0(\mathcal{O}(K-L))=0}}{h^2(\tilde{S}, \mathcal{O}(L))}$$

$$= \dim|L| + 1 - h^1(\mathcal{O}_{\tilde{S}}(L))$$

$$\Rightarrow \quad \dim|L| = \frac{n^2+3n}{2} - d + h^1(\mathcal{O}_{\tilde{S}}(L)) = \frac{n^2+3n}{2} - d + \omega$$

$\Rightarrow \omega = h^1(\mathcal{O}_{\tilde{S}}(L))$. by comparing with Italian notations.

It remains to show that $\dim|\mathcal{J}_{\Gamma_0}(n)| = \dim|L|$.
Earlier, we showed $|L| = \{\pi^*C - E\}$, $C \in |\mathcal{J}_{\Gamma_0}(n)|$
$\Rightarrow \dim\{\pi^*C - E\} = \dim\{\pi^*C\} = \dim\{C\} = \dim|\mathcal{J}_{\Gamma_0}(n)|$ by considering π^* is one to one generically.
$\Rightarrow \dim|L| = \dim|\mathcal{J}_{\Gamma_0}(n)| = r.$ $\square$

It is now clear that $\omega = 0$ if and only if the points in Γ_0 impose independent conditions on the linear system $|\mathcal{O}_{\mathbb{P}^2}(n)|$. As was seen in the Cayley – Bacharach theorem in Section 2 of this chapter, it may well happen that $h^1(\mathcal{J}_{\Gamma_0}(n)) > 0$ – indeed, this is frequently the interesting case.