$$\Rightarrow \bar\partial_M *_{M\times N} \bar\partial_N *_{M\times N}(\varphi\wedge\psi) = \bar\partial_M *_{N\times N} \bar\partial_N \in$$
$$= \bar\partial_M *_{M\times N} \in (-1)^{(m-p+m-q)}(*_M\varphi\wedge\bar\partial_N *_N\psi)$$
$$= \in (-1)^{p+q}\, \bar\partial_M *_{M\times N}(*_M\varphi\wedge\bar\partial_N *_N\psi) = \in (-1)^{p+q}(-1)^{(m-p+m-q)(p'+q'-1)}\bar\partial_M *_M *_M\varphi\wedge *_N\bar\partial_N *_N\psi$$
$$= \in (-1)^{p+q}(-1)^{(p+q)(p'+q'-1)}\bar\partial_M(*_M^2\varphi)\wedge *_N\bar\partial_N *_N\psi$$
$$= \bar\partial_M(*_M^2\varphi)\wedge *_N\bar\partial_N *_N\psi. \quad \cdots\quad (1)$$

On the other hand, $*_{M\times N}\bar\partial_N *_{M\times N}\bar\partial_M(\varphi\wedge\psi) = *_{M\times N}\bar\partial_N *_{M\times N}(\bar\partial_M\varphi\wedge\psi)$

$$= *_{M\times N}\bar\partial_N(*_M\bar\partial_M\varphi\wedge *_N\psi)(-1)^{(p+q+1)(p'+q')}$$
$$= *_{M\times N}(*_M\bar\partial_M\varphi\wedge\bar\partial_N *_N\psi)(-1)^{(p'+q')(p+q+1)}(-1)^{2m-p-q-1}$$
$$= *_M^2\bar\partial_M\varphi\wedge *_N\bar\partial_N *_N\psi\,(-1)^{(p'+q')(p+q+1)+p+q+1}(-1)^{(2m-p-q-1)(2n-p'-q'+1)}$$

$$= (-1)^0 *_M^2\bar\partial_M\varphi\wedge *_N\bar\partial_N *_N\psi = *_M^2\bar\partial_M\varphi\wedge *_N\bar\partial_N *_N\psi. \cdots (2)$$

φ is (p,q) type $\Rightarrow *_M^2 = (-1)^{p+q}$
$\bar\partial_M\varphi$ is $(p,q+1)$ type $\Rightarrow *_M^2 = (-1)^{p+q+1}$.

$$\Rightarrow (1) = (-1)^{p+q}\bar\partial_M\varphi\wedge *_N\bar\partial_N *_N\psi$$
$$(2) = (-1)^{p+q+1}\bar\partial_M\varphi\wedge *_N\bar\partial_N *_N\psi$$
$$\Rightarrow (1) + (2) = 0.$$

⌟)

These relations imply that
$$\Delta_{M\times N} = \Delta_M + \Delta_N.$$
More precisely, on decomposable forms

$$\Delta_{M\times N}(\psi\otimes\eta) = (\Delta_M\psi)\otimes\eta + \psi\otimes\Delta_N\eta$$
and by the lemma this determines $\Delta_{M\times N}$ on all forms.

⌜ $\Delta_{M\times N} = \bar\partial_{M\times N}^*\bar\partial_{M\times N} + \bar\partial_{M\times N}\bar\partial_{M\times N}^* = (\bar\partial_M^* + \bar\partial_N^*)(\bar\partial_M + \bar\partial_N) + (\bar\partial_M + \bar\partial_N)$
$(\bar\partial_M^* + \bar\partial_N^*) = \bar\partial_M^*\bar\partial_M + \bar\partial_N^*\bar\partial_N + \cancel{\bar\partial_M^*\bar\partial_N} + \cancel{\bar\partial_N^*\bar\partial_M} + \bar\partial_M\bar\partial_M^* + \bar\partial_N\bar\partial_N^*$
$+ \cancel{\bar\partial_M\bar\partial_N^*} + \cancel{\bar\partial_N\bar\partial_M^*} = \Delta_M + \Delta_N$
where $\Delta_M = \bar\partial_M^*\bar\partial_M + \bar\partial_M\bar\partial_M^*$ $\Delta_N = \bar\partial_N^*\bar\partial_N + \bar\partial_N\bar\partial_N^*$.