$\Rightarrow$ By the argument above,

$$\frac{\varphi_{(I-L_j)\cup\{j\},K}}{z_j} = \frac{\varphi_{J,K}}{z_j} \text{ is holomorphic for each } z_j.$$

$\Rightarrow$ Since $z_1, \cdots z_j \cdots$ ($j \in J$) are irreducible and $\mathcal{O}_n$ is UFD, $\dfrac{\varphi_{J,K}}{z_1 \cdots z_j \cdots} = \dfrac{\varphi_{JK}}{z_J}$ is holomorphic.

($\because$ φ_{JK} is divided by each z_j)

This implies that

$$\varphi = \sum \frac{\varphi_{JK}}{z_J} \frac{dz_{I-J}}{z_{I-J}} \wedge dz_K$$ is expressed by holomorphic forms and the logarithmic differentials dz_i/z_i.

$\square$

Intuitively, if φ contains a term with $1/z_i$ but no dz_i in the numerator, then $d\varphi$ will contain dz_i/z_i^2 - what we have verified is that no cancellation occurs.

The main local result, which as we will see plays the role of a Poincare lemma in the present context, is the following

Lemma. The two inclusions

$$\begin{cases} \Omega^*(\log D) \subset \mathcal{A}^*(*D), \\ \Omega^*(*D) \subset \mathcal{A}^*(*D), \end{cases}$$

are both quasi-isomorphisms.

Proof. At a point $p \notin D$, the stalks are