The resulting surface S, by the last lemma of the previous section, is isomorphic to $\mathbb{P}^2$; so we have given, up to an automorphism of $\mathbb{P}^2$, a birational map φ_{abc} of $\mathbb{P}^2$ to itself.

Γ

$$\begin{tikzcd} & \widetilde{\mathbb{P}}^2 \arrow[dl, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^2 & & M \end{tikzcd}$$

$$H_i(\widetilde{\mathbb{P}}^2) = H_i(M) \oplus H_i(\widetilde{L}_{ab}) \oplus H_i(\widetilde{L}_{bc}) \oplus H_i(\widetilde{L}_{ac})$$
$$= H_i(M) \oplus H_i(\mathbb{P}^1) \oplus \cdot \oplus H_i(\mathbb{P}^1)$$
$$\Rightarrow H_1(\widetilde{\mathbb{P}}^2) = H_1(M) = 0$$
$H_2(\widetilde{\mathbb{P}}^2) = H_2(M) \oplus \mathbb{C}^3 = \mathbb{C}^4$, since $H_2(\widetilde{\mathbb{P}}^2) = H_2(\mathbb{P}^2) \oplus \mathbb{C}^3$. $\Rightarrow b^1(M) = 1 = b^3(M)$

$b^0(M) = b^4(M) = 1 \qquad b^1(M) = b^3(M) = 0$.

$$K_{\widetilde{\mathbb{P}}^2} = \pi'^* K_M + \widetilde{L}_{ab} + \widetilde{L}_{bc} + \widetilde{L}_{ac}$$
$$\Rightarrow K_M \cdot D = \pi'^* D \cdot \pi'^* K_M$$
$$= \pi'^* D \cdot (K_{\widetilde{\mathbb{P}}^2} - \widetilde{L}_{ab} - \widetilde{L}_{bc} - \widetilde{L}_{ac})$$
$= \pi'^* D \cdot K_{\widetilde{\mathbb{P}}^2} \leq 0$ (?) for any divisor $D \subset M$.

To show this,

since $K_{\widetilde{\mathbb{P}}^2} = \pi^* K_{\mathbb{P}^2} + E_a + E_b + E_c$,

$(\pi^* D' + (n_1 E_a + n_2 E_b + n_3 E_c)) \cdot K_{\widetilde{\mathbb{P}}^2}$

$= D' \cdot K_{\mathbb{P}^2} - n_1 - n_2 - n_3 \leq 0$ for $n_1, n_2, n_3 \geq 0$ and $D' \subset \mathbb{P}^2$ divisor.

$\Rightarrow K_{\widetilde{\mathbb{P}}^2}$ is not positive $\Rightarrow \pi'^* D \cdot K_{\mathbb{P}^2} \leq 0 \Rightarrow K_M$ is not positive. $\Rightarrow$ By the lemma on P487, $M \cong \mathbb{P}^2$.

$$\begin{tikzcd} & \widetilde{\mathbb{P}}^2 \arrow[dl, shift right, "\pi"'] \arrow[dr, "\pi'"] & \\ \mathbb{P}^2 \arrow[ur, shift right, "\pi^{-1}"'] & & M \end{tikzcd}$$

$\Rightarrow \pi' \circ \pi^{-1} : \mathbb{P}^2 \longrightarrow M$ is a birational map with an inverse