$r = g - hf \in I$ and since $r \in \mathcal{O}_{n-1}[z_n]$, $r \in I \cap \mathcal{O}_{n-1}[z_n] = I'$ $\Rightarrow$ $g = hf + r = hf + a_1 f_1 + \cdots + a_k f_k \in \{f, f_1 \cdots f_k\}$.

Q.E.D.

Our discussion of commutative algebra will center around $\mathcal{O}$-modules, usually denoted by $M, N, R, \cdots$ and which we always assume to be finitely generated. Choosing generators $m_1, \cdots m_k$ for an $\mathcal{O}$-module M, there is an exact sequence

$$0 \longrightarrow R \longrightarrow \mathcal{O}^{(k)} \xrightarrow{\pi} M \longrightarrow 0$$

of $\mathcal{O}$-modules, where

$$\mathcal{O}^{(k)} = \underbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}_{k}$$

is the free $\mathcal{O}$-module of rank k,

$$\pi(g_1, g_2 \cdots g_k) = g_1 m_1 + \cdots + g_k m_k,$$

and

$$R = \{(g_1 \cdots g_k) : g_1 m_1 + \cdots + g_k m_k = 0\}$$

is the module of relations among the m_i's. We claim that R is again finitely generated. The proof is by induction on k, with the case $k = 1$ being that of an ideal in $\mathcal{O}$ just discussed.

$k = 1$, $\mathcal{O} \xrightarrow{\pi} M$, $g \longmapsto gm$ $\Rightarrow$ $R = \{g \in \mathcal{O} \mid gm = 0\}$.

$\Rightarrow$ $\forall g' \in \mathcal{O}$ $g'g'' \in R$ for $g'' \in R$, since $g'g''m = 0$.

$\Rightarrow$ R is an ideal of $\mathcal{O}$ which is an $\mathcal{O}$-module. $\Rightarrow$ Since $\mathcal{O}$ is Noetherian, R is finitely generated.