For small ϵ, $D(\epsilon)$ is just a tubular nbd around V in M, and

$$\int_M \Theta \wedge \psi = \lim_{\epsilon\to 0} 2\pi i \int_{M-D(\epsilon)} dd^c \log|S|^2 \wedge \psi$$

$$= \lim_{\epsilon\to 0} \frac{2\pi}{i} \int_{\partial D(\epsilon)} d^c \log|S|^2 \wedge \psi$$

by Stoke's theorem.

$\ulcorner$ $d(d^c \log|S|^2 \wedge \psi) = dd^c \log|S|^2 \wedge \psi$ since $d\psi = 0$.

$\int_M \Theta\wedge\psi = \lim_{\epsilon\to 0}\int_{M-D(\epsilon)} \Theta\wedge\psi$ since D is measure zero.

$\lim_{\epsilon\to 0}\int_{D(\epsilon)} \Theta\wedge\psi = 0$ $\lrcorner$

In $U_\alpha \cap D(\epsilon)$, write

$|S|^2 = |f_\alpha|^2 h_\alpha = f_\alpha \cdot \bar{f}_\alpha \cdot h_\alpha$ with $h_\alpha > 0$; we have

$$d^c \log|S|^2 = d^c \log(f_\alpha \cdot \bar{f}_\alpha \cdot h_\alpha)$$

$$= \frac{i}{4\pi}\left(\bar\partial \log \bar{f}_\alpha - \partial \log f_\alpha + (\bar\partial - \partial)\log h_\alpha\right).$$

$\ulcorner$

$$\begin{tikzcd}
{[D]|_{U_\alpha}} \arrow[rr, "\varphi_\alpha"] & & U_\alpha \times \mathbb{C} \\
S_\alpha(x) = S|_{U_\alpha}(x) & & (x, f_\alpha(x)) \\
& U_\alpha \ni x \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

Consider a holomorphic map from $U_\alpha \longrightarrow \mathbb{C}$ defined by $x \longmapsto 1$. $\Rightarrow$ Let $\bar{S}_\alpha$ be the corresponding map of $x \longmapsto 1$ from $U_\alpha \longrightarrow [D]|_{U_\alpha}$.

$\Rightarrow S_\alpha = f_\alpha \cdot \bar{S}_\alpha \Rightarrow \langle S_\alpha, S_\alpha\rangle = |f_\alpha|^2 \langle \bar{S}_\alpha \bar{S}_\alpha\rangle$

Let $h_\alpha = \langle \bar{S}_\alpha \bar{S}_\alpha\rangle$, which is nonvanishing. on U_α.