$v^*(v)=0$ for all $v^* \in \Lambda^\perp \}$

Lemma. W is the minimal subspace of V such that Λ is in the image of $\Lambda^k W \longrightarrow \Lambda^k V$.

Proof. Let $w_1, w_2, \cdots w_\ell$ be a basis for W, and complete it by $u_{\ell+1}, \cdots u_n$ to a basis for V. Denote the dual basis of V^* by $\{w_i^*, u_\alpha^*\}$.

$w_i^*(w_j) = \delta_{ij}$. $w_i^*(u_\alpha) = 0$. $u_\alpha^*(u_\beta) = \delta_{\alpha\beta}$

$u_\alpha^*(w_j) = 0$.

Setting $\mathcal{U} = \mathbb{C}\{u_{\ell+1}, \cdots u_n\}$, the direct sum decomposition $V = W \oplus \mathcal{U}$ induces

$$\Lambda^k V \cong \Lambda^k W \oplus (\Lambda^{k-1} W \otimes \mathcal{U}) \oplus (\Lambda^{k-2} W \otimes \Lambda^2 \mathcal{U}) \oplus \cdots .$$

We want to show that Λ lies in the first factor. Write the component of Λ in the second factor as $\sum_{\alpha=\ell+1}^{n} \Lambda_\alpha \otimes u_\alpha$, where $\Lambda_\alpha \in \Lambda^{k-1} W$. Since

$$i(u_\alpha^*) : \Lambda^{k-m} W \otimes \Lambda^m \mathcal{U} \longrightarrow \Lambda^{k-m} W \otimes \Lambda^{m-1} \mathcal{U}$$

and $i(u_\alpha^*)\Lambda = 0$, we deduce that all $\Lambda_\alpha = 0$.

$u_\alpha^* \in V^*$ & $u_\alpha^* \in \mathcal{U}$.

$\Rightarrow$ $i(u_\alpha^*) : \Lambda^m \mathcal{U} \longrightarrow \Lambda^{m-1} \mathcal{U}$

We have to show that $i(u_\alpha^*) : \Lambda^k V \longrightarrow \Lambda^{k-1} V$ is $\mathrm{id} \otimes i(u_\alpha^*)$ on $\Lambda^{k-m} W \otimes \Lambda^m \mathcal{U}$.

$$\begin{array}{ccc} \Lambda^k V & \xrightarrow{i(u_\alpha^*)} & \Lambda^{k-1} V \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \bigoplus_{m=0}^{k} (\Lambda^{k-m} W \otimes \Lambda^m \mathcal{U}) & \longrightarrow & \bigoplus_{m=0}^{k} \Lambda^{k-m} W \otimes \Lambda^{m-1} \mathcal{U} \end{array}$$