$\Rightarrow$ $\dfrac{S_1}{S_2}$ may be considered as a holomorphic function from $M-B$ to $\mathbb{P}^1$. We may express exactly as follows:

$$\begin{array}{ccc} M-B & \xrightarrow{\quad f \quad} & \mathbb{P}^1 \\ \Psi & & \\ p & \longmapsto & [(S_{1,\alpha}(p),\ S_{2,\alpha}(p))] . \qquad p \in U_\alpha \end{array}$$

where. if $\{(\varphi_\alpha, U_\alpha)\}$ is a trivializations of $L=[D]$,

$$\begin{tikzcd} L_{U_\alpha} \arrow[r, "\varphi_\alpha"] & U_\alpha \times \mathbb{C} \\ & U_\alpha \arrow[ul, "S_1"] \arrow[u, "{(x,\ S_{1,\alpha}(x))}"'] \end{tikzcd} \qquad \begin{tikzcd} L_{U_\alpha} \arrow[r, "\varphi_\alpha"] & U_\alpha \times \mathbb{C} \\ U_\alpha \arrow[u, "S_2"] \arrow[ur, "{(x,\ S_{2,\alpha}(x))}"'] & \end{tikzcd}$$

Since $g_{\alpha\beta} S_{1,\beta}(x) = S_{1,\alpha}(x)$ $\qquad g_{\alpha\beta} S_{2,\beta}(x) = S_{2,\alpha}(x)$,

f is well-defined globally and holomorphic.

Suppose f is singular at p. & suppose $S_{2,\alpha}(p) \neq 0$.

$$\frac{\partial \dfrac{S_{1,\alpha}}{S_{2,\alpha}}}{\partial z_i}(p) = 0 = \frac{\dfrac{\partial S_{1,\alpha}}{\partial z_i} S_{2,\alpha} - \dfrac{\partial S_{2,\alpha}}{\partial z_i} S_{1,\alpha}}{(S_{2,\alpha})^2}(p) = 0$$

$$\Rightarrow \quad \frac{\partial S_{1,\alpha}}{\partial z_i}(p)\, S_{2,\alpha}(p) = \frac{\partial S_{2,\alpha}}{\partial z_i}(p)\, S_{1,\alpha}(p)$$

$$\Rightarrow \quad \frac{\dfrac{\partial S_{1,\alpha}}{\partial z_i}(p)}{\dfrac{\partial S_{2,\alpha}}{\partial z_i}(p)} = \frac{S_{1,\alpha}(p)}{S_{2,\alpha}(p)} \qquad \text{if } \frac{\partial S_{2,\alpha}}{\partial z_i}(p) \neq 0$$

$$\Rightarrow \quad \frac{\partial S_{1,\alpha}}{\partial z_i}(p) - \frac{S_{1,\alpha}(p)}{S_{2,\alpha}(p)} \frac{\partial S_{2,\alpha}}{\partial z_i}(p) = 0$$