$$\sum_{\xi} |\xi|^{2s+4} |u_\xi|^2 < \infty. \quad \Leftrightarrow \quad \sum_{\xi} (1+|\xi|^2)^{s+2} |u_\xi|^2 < \infty$$

since $1+|\xi|^2 < 2|\xi|^2$. and $|\xi|^2+1 > |\xi|^2$.

Set $v = \sum_{\xi \neq 0} i\xi u_\xi e^{i\xi x}$.

$$\Rightarrow \quad \|v\|_{s+1}^2 = \sum_{\xi} (1+|\xi|^2)^{s+1} |v_\xi|^2 = \sum_{\xi} (1+|\xi|^2)^{s+1} |\xi u_\xi|^2$$

$$\leq \sum (2|\xi|^2)^{s+1} |\xi|^2 |u_\xi|^2 = 2^{s+1} \sum |\xi|^{2s+4} |u_\xi|^2 < \infty.$$

$\Rightarrow$ $v \in H_{s+1}$, and therefore. $v \in C^s(T)$ by induction hypothesis.

The convergence being uniform, we may integrate term by term:

$$\int_0^x v(t)\,dt = \int_0^x \sum i\xi u_\xi e^{i\xi t}\,dt$$

$$= \sum i\xi u_\xi \int_0^x e^{i\xi t}\,dt = \sum_{\xi} \cancel{i\xi} u_\xi \left.\frac{e^{i\xi t}}{\cancel{i\xi}}\right]_0^x = \sum_{\xi} u_\xi (e^{i\xi x} - 1)$$

$$= \sum_{\xi \neq 0} u_\xi e^{i\xi x} - \sum_{\xi \neq 0} u_\xi = u(x) - u_0 - (u(0) - u_0)$$

$$= u(x) - u(0).$$

$\Rightarrow$ $u'(x) = v(x)$. and $u \in C^{s+1}(T)$.

Γ $n=2$ case. $u = \sum_{\xi \in \mathbb{Z}^2} u_\xi e^{i\langle \xi, x\rangle}$

$$v_1 = \sum_{\xi_1 \neq 0} i\xi_1 u_\xi e^{i\langle \xi, x\rangle}, \quad v_2 = \sum_{\xi_2 \neq 0} i\xi_2 u_\xi e^{i\langle \xi, x\rangle}$$

$$\Rightarrow \frac{\partial u}{\partial x_i} = \sum_{\xi \in \mathbb{Z}^2} u_\xi (i\xi_i) e^{i\langle \xi, x\rangle}$$

$\lrcorner$

Q.E.D.