Indeed, we see from this that the identification

$$E \longrightarrow \mathbb{P}(T_x(M))$$

given by

$$(0, \ell) \longmapsto \left[\sum \ell_i \frac{\partial}{\partial z_i} \right]$$

is likewise independent of the coordinate system chosen.

$\sqcap$
$$\tilde{\Delta} - E \xleftarrow{z^{-1}} U - \{x\} \xleftarrow{z} \tilde{\Delta} - E .$$

$$(0, \ell) \longleftarrow (0, \ell)$$

through $z^{-1*}(e_i) = \frac{\partial}{\partial z_i}$

$\Rightarrow$ Since e_i in $T_0\Delta$ corresponds to $\frac{\partial}{\partial z_i}$ in $T_x M$, the pull-back z^{-1*} of $(0, \ell)$ is

$$\sum \ell_i \frac{\partial}{\partial z_i} .$$

If $\{z_i' = f_i(z)\}$ are other charts in Δ,

$$(0, \ell') \longmapsto \left[\sum \ell_i' \frac{\partial}{\partial z_i'} \right] .$$

$\Rightarrow$ $\frac{\partial}{\partial z_i} = \sum_\ell \frac{\partial f_\ell}{\partial z_i}(0) \frac{\partial}{\partial z_\ell'}$ from $\frac{\partial \psi}{\partial z_i} = \frac{\partial \psi(z_1' \cdots z_n')}{\partial z_i} =$

$$\left(\frac{\partial \psi}{\partial z_1'} \cdots \frac{\partial \psi}{\partial z_n'} \right) \cdot \left(\frac{\partial f_1}{\partial z_i}, \frac{\partial f_2}{\partial z_i} \cdots \frac{\partial f_n}{\partial z_i} \right)$$

$$\begin{array}{ccccc}
(0, \ell) & \longmapsto & \sum \ell_i \frac{\partial}{\partial z_i} & = & \sum \ell_i \frac{\partial f_j}{\partial z_i}(0) \frac{\partial}{\partial z_j'} \\
E & \longrightarrow & \mathbb{P}(T_x(M)) & & \\
\downarrow & & \| & & \| \\
E' & \longrightarrow & \mathbb{P}(T_x(M)) & & \\
(0, \ell') & \longmapsto & \sum \ell_i' \frac{\partial}{\partial z_i'} & = & \sum \frac{\partial f_i}{\partial z_j}(0) \ell_j \frac{\partial}{\partial z_i'}
\end{array}$$

independent of the choice of coordinate system

$\lrcorner\!\lrcorner$