Date

Special Linear Systems I.

The Riemann–Roch formula describes exactly the behavior of the "generic" linear system on a Riemann surface, but it does not tell us much about irregular linear systems. We will now try to fill in the gap with some classical theorems relating the dimension, degree, and genus of special linear systems. Our basic lemma is

Lemma. For $C \subset \mathbb{P}^n$ a nondegenerate curve, the points of a generic hyperplane section of C are in general position; i.e., no n of them are linearly dependent.

Proof. Suppose C has degree d, and let $H_0 \subset \mathbb{P}^n$ be a hyperplane meeting C in d distinct points $p_1, \dots, p_d$. Then for H in a sufficiently small neighborhood U of H_0 in $\mathbb{P}^{n*}$, the points $\{p_i(H)\}$ of intersection of H with C will vary holomorphically with $H \in \mathbb{P}^{n*}$.

We can get a hyperplane H_0 s.t. $H_0 \cap C$ is a set of distinct points by Bertini's theorem.

Consider $G(n, n+1) \cong \mathbb{P}^{n*}$. $\Rightarrow$ $H_0 \in G(n, n+1)$.

$\Rightarrow$ Around H_0, we may express H as $(a_0 Z_0 + \cdots + a_n Z_n = 0)$, where $(a_0' Z_0 + \cdots + a_n' Z_n = 0) = H_0$ and H varies holomorphically with $(a_0, \cdots, a_n) \in \mathbb{C}^{n+1}$.