We start by noting that if the flag $\{V_\alpha\}$ is generically chosen, each subspace $\bar{V}_\alpha \subset \mathbb{P}^{n+1}$ will intersect F in a smooth $(\alpha-2)$-dimensional quadric $F_{\alpha-2}$.

$\ulcorner$ $\bar{V}_\alpha = \mathbb{P}^\alpha \Rightarrow \mathbb{P}^{\alpha-1} \cap F = \bar{V}_\alpha \cap F$ has the dimension $n-1+\alpha-(n+1)$. $\Rightarrow$ $\alpha-2$ is the dimension of $\mathbb{P}^\alpha \cap F$. $\lrcorner$

Now, as we have seen, $F_{\alpha-2}$ can not contain any linear subspaces of projective dimension greater than $(\alpha-2)/2$; thus, no k-plane Λ_k lying on F can meet $\bar{V}_\alpha$ in a space of projective dimension $> (\alpha-2)/2$.

$\ulcorner$ Think simply. If some k-plane Λ_k lying on F meets $\bar{V}_\alpha$ in a space of projective dimension $>$ $(\alpha-2)/2$, then $\Lambda_k \cap \bar{V}_\alpha$ is of dimension $> (\alpha-2)/2$.
$\Rightarrow$ $F_{\alpha-2}$ contains a projective space of dim $> (\alpha-2)/2$, which is absurd. $\Lambda_k \cap \bar{V}_\alpha \subset F \cap \bar{V}_\alpha = F_{\alpha-2}$. $\lrcorner$

If $\#(\Sigma_{k,n} \cdot \sigma_b)$ is to be nonzero, then we must have

$$n-k+1+i-b_i \geq 2i \quad \text{for all } i;$$

i.e.,

$$b_i \leq n-k-i+1.$$