Proposition. All eigenvalues for H are integers, and we can write
$$V = V_n \oplus V_{n-2} \oplus V_{n-4} \oplus \cdots \oplus V_{-n}.$$

pf) Let v be primitive, and suppose
$$Y^n v \neq 0, \qquad Y^{n+1} v = 0, \qquad \text{and} \qquad Hv = \lambda v.$$
Then
$$\begin{aligned} Xv &= 0, \\ XYv &= YXv + Hv = \lambda v \\ XY^2 v &= YXYv + HYv = Y\lambda v + (\lambda-2)Yv \\ &= (2\lambda - 2) Yv, \end{aligned}$$
and in general
$$XY^m v = YXY^{m-1}v + HY^{m-1}v, \text{ so we have}$$

$$\begin{aligned} XY^m v &= (\lambda + (\lambda-2) + (\lambda-4) + \cdots + (\lambda - 2(m-1)))\, Y^{m-1} v \\ &= (m\lambda - m^2 + m)\, Y^{m-1} v, \end{aligned}$$
and since $Y^n v \neq 0$, $Y^{n+1} v = 0$.
$$XY^{n+1} v = ((n+1)\lambda - (n+1)^2 + n+1)\, Y^n v = 0$$
$$\Rightarrow \quad \lambda - (n+1) + 1 = 0 \quad \Rightarrow \quad \lambda = n, \qquad \text{Q.E.D.}$$

In summary, the irreducible sl_2 modules are indexed by nonnegative integers n; for each such n, the corresponding sl_2-module $V(n)$ has dimension $n+1$. Explicitly,

$V(n) \cong Sym^n(\mathbb{C}^2)$ 「 ${}_2H_n = {}_{n+1}C_n = n+1 = {}_{2+n-1}C_n$ 」

is the n-th symmetric power of the vector space $\mathbb{C}^2$. The eigenvalues of H acting on $V(n)$ are $-n, -n+2, \cdots, n-2, n$, each appearing with multiplicity 1.

For any sl_2-module V, not necessarily irreducible, we define the Lefschetz decomposition of V as follows: let $PV = \mathrm{Ker}(X)$; then
$$V = PV \oplus YPV \oplus Y^2 PV \oplus \cdots, \quad \text{and}$$