$$= \sum x_i^2\, dx_1 \wedge \cdots \wedge dx_n = r^2\, dx_1 \wedge \cdots \wedge dx_n.$$

(Since, if $r\,dr \wedge a\, dx_{i_1} \wedge \cdots \wedge dx_{i_{n-1}} = 0$, $a = 0$, ⇒ Nonsense.!)

Anyway, $* r\,dr = \sum \Phi_i(x)$ can be proved by computation easily.

This form σ belongs to $L^{n-1}(\mathbb{R}^n, loc)$, is invariant under the proper orthogonal group, and is smooth on $\mathbb{R}^n - \{0\}$.

Pf σ is $(n-1)$-form since $\sigma = C_n \dfrac{1}{r^n} \sum \Phi_i(x) = \dfrac{C_n}{r^n} \sum (-1)^{i-1}$ $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$. The coefficients $\dfrac{(-1)^{i-1}}{r^n} C_n$ are locally integrable.

Let's try for $n=2$.

$$\Rightarrow \sum \Phi_i = x_1\, dx_2 - x_2\, dx_1$$

Let $x_1 = a_{11} y_1 + a_{21} y_2$, $x_2 = a_{12} y_1 + a_{22} y_2$.

$$\Rightarrow dx_1 = a_{11}\, dy_1 + a_{21}\, dy_2, \quad dx_2 = a_{12}\, dy_1 + a_{22}\, dy_2$$

$$\Rightarrow (a_{11} y_1 + a_{21} y_2)(a_{12}\, dy_1 + a_{22}\, dy_2) - (a_{12} y_1 + a_{22} y_2)(a_{11}\, dy_1 + a_{21}\, dy_2)$$

$$= dy_1 (a_{11} a_{12} y_1 + a_{21} a_{12} y_2 - a_{11} a_{12} y_1 - a_{11} a_{22} y_2)$$

$$+ dy_2 (a_{11} y_1 a_{22} + a_{21} a_{22} y_2 - a_{12} a_{21} y_1 - a_{22} a_{21} y_2)$$

$$= dy_1 \{ (a_{11} a_{12} - a_{11} a_{12}) y_1 + (a_{21} a_{12} - a_{11} a_{22}) y_2 \}$$

$$+ dy_2 \{ (a_{11} a_{22} - a_{12} a_{21}) y_1 + (a_{21} a_{22} - a_{22} a_{21}) y_2 \}$$

⇒ If $a_{11} a_{12} - a_{11} a_{12} = 0$, $a_{21} a_{12} - a_{11} a_{22} = 1$, $a_{12} a_{21} - a_{11} a_{22} = 1$

$a_{21} a_{22} - a_{22} a_{21} = 0$, then $\sum \Phi_i$ is invariant.

$$\begin{pmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{pmatrix} \begin{pmatrix} -a_{22}, & a_{12} \\ a_{21}, & -a_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since $d\Phi_i(x) = \Phi(x)$, it follows that in $\mathbb{R}^n - \{0\}$

$$d\sigma = C_n \left(\frac{n\,\Phi(x)}{r^n} - \frac{n\, r\, dr \wedge *(r\, dr)}{r^{n+2}} \right) = 0.$$