Date

canonical mapping of S. $\iota_K(S) \subset \mathbb{P}^{g-1}$ the canonical curve of S.

Now, the map ι_K is 1-1 if for any points $p, q \in S$, we can find a $\omega \in H^0(S, \Omega^1)$ with $\omega(p) = 0$, $\omega(q) \neq 0$; it is an immersion if for any $p \in S$ there exists ω vanishing exactly to order 1 at p.

Γ
$$\iota_K : S \longrightarrow \mathbb{P}^2$$
$$p \longmapsto [\omega_1(p), \omega_2(p), \omega_3(p)]$$

Locally, $\omega_1 = h_1 dz$, $\omega_2 = h_2 dz$, $\omega_3 = h_3 dz$.
Assume. $h_1 = z\, g(z)$, $g(0) \neq 0$ and $\omega_3 \neq 0$ at 0.

$$\Rightarrow \quad z \longmapsto \left(\frac{h_1}{h_3}, \frac{h_2}{h_3} \right).$$

$$\frac{\partial \frac{h_1}{h_3}}{\partial z} = \frac{\frac{\partial h_1}{\partial z} \cdot h_3 - \frac{\partial h_3}{\partial z} h_1}{h_3^2} \neq 0 \quad \text{at } z = 0.$$

$\Rightarrow$ ι_K is an immersion. ⅃

Thus, ι_K is an imbedding if and only if for any points p, q, not necessarily distinct,

$$h^0(K - p - q) < h^0(K - p) = g - 1.$$

Γ (⇒) ① p, q distinct $H^0(S, \Omega^1) \ni \omega$ with $\omega(p) = 0$ and $\omega(q) \neq 0$. Consider the following map

$$H^0(S, \Omega^1(-p-q)) \xrightarrow{\phi} H^0(S, \Omega^1(-p))$$
$$\sigma \longmapsto \sigma/s_0 \quad \text{where } (s_0 = 0) = -q.$$