orthogonal to the normal dr to spheres, and that has integral 1 over a sphere of any radius.

⌈ At this point, $\int_{\|x\|=\epsilon} \sigma = 1$ is the only thing to be sure, the rest is not clear. Let's see later. ⌋

In $\mathbb{R}^2$ with coordinates $(x, y) = (r\cos\theta, r\sin\theta)$,

$$\sigma = \frac{1}{2\pi} \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{1}{2\pi} d\theta .$$

⌈
$$y = r\sin\theta, \qquad dy = \sin\theta\, dr + r\cos\theta\, d\theta$$
$$x = r\cos\theta, \qquad dx = \cos\theta\, dr - r\sin\theta\, d\theta.$$
$$x\,dy - y\,dx = r\cos\theta\,(\sin\theta\, dr + r\cos\theta\, d\theta) - r\sin\theta\,(\cos\theta\, dr - r\sin\theta\, d\theta) = r^2\cos^2\theta\, d\theta + r^2\sin^2\theta\, d\theta = r^2 d\theta$$
$$\Rightarrow \sigma = \frac{1}{2\pi} d\theta .$$
⌋

In general if $x = r\omega$, where $r = \|x\|$ and $\omega \in S^{n-1}$ are polar coordinates in $\mathbb{R}^n$ then we may write

$$\sigma = C_n\, d\omega .$$

For $\varphi \in C_c^\infty(\mathbb{R}^n)$, by Stokes' theorem,

$$-\int_{\mathbb{R}^n} d\varphi \wedge \sigma = \lim_{\epsilon\to 0} -\int_{\mathbb{R}^n - \{\|x\|\leq\epsilon\}} d\varphi\wedge\sigma = \lim_{\epsilon\to 0}\int_{\|x\|=\epsilon} \varphi\sigma$$
$$= \varphi(0).$$

⌈
$$\int_{\mathbb{R}^n - \{\|x\|\leq\epsilon\}} d\varphi\wedge\sigma = \int_{\mathbb{R}^n - \{\|x\|\leq\epsilon\}} d(\varphi\wedge\sigma) \quad \text{since } d\sigma = 0.$$
$$= -\int_{\|x\|=\epsilon} \varphi\wedge\sigma = -\int_{\|x\|=\epsilon}\varphi\sigma$$
by Stokes' theorem, here we used the fact $\varphi \in C_c^\infty(\mathbb{R}^n)$ to make the integral make sense.