The long exact sequence of Ext's gives

$$\longrightarrow \mathrm{Ext}_{\mathcal{O}}^{k}(\mathcal{O}/I', \mathcal{O}) \longrightarrow \mathrm{Ext}_{\mathcal{O}}^{k}(I/I', \mathcal{O}) \longrightarrow \mathrm{Ext}^{k+1}(\mathcal{O}/I, \mathcal{O}) \to \cdots$$

Our desired injectivity thus follows from

$$\mathrm{Ext}_{\mathcal{O}}^{r-1}(I/I', \mathcal{O}) = 0,$$

which we now prove.

⊢ $\mathrm{Ext}_{\mathcal{O}}^{r-1}(I/I', \mathcal{O}) \longrightarrow \mathrm{Ext}_{\mathcal{O}}^{r}(\mathcal{O}/I, \mathcal{O}) \longrightarrow \mathrm{Ext}_{\mathcal{O}}^{r}(\mathcal{O}/I', \mathcal{O}) \longrightarrow$

$\qquad \overset{\shortparallel}{0}$

We know that $\mathrm{Ext}_{\mathcal{O}}^{r}(\mathcal{O}/I, \mathcal{O}) \cong \mathcal{O}/I$ &

$\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel\, H_0(E.(f))$

$$\begin{array}{ccc}
\mathrm{Ext}_{\mathcal{O}}^{r}(\mathcal{O}/I, \mathcal{O}) & \longrightarrow & \mathrm{Ext}_{\mathcal{O}}^{r}(\mathcal{O}/I', \mathcal{O}) \\
\shortparallel & & \shortparallel \\
H^r(\mathrm{Hom}(E.(f), \mathcal{O})) & & H^r(\mathrm{Hom}(E.(f'), \mathcal{O})) \\
\Psi & & \Psi \\
p & \longmapsto & \Delta \cdot p
\end{array}$$

For, $\quad \begin{array}{ccc} E_r' & \xrightarrow{\lambda} & E_r \\ \Psi & & \Psi \\ g & \longmapsto & \Delta \cdot g \end{array} \quad \Rightarrow \quad \begin{array}{ccc} \mathrm{Hom}(E_r, \mathcal{O}) & \xrightarrow{\lambda^*} & \mathrm{Hom}(E_r', \mathcal{O}) \\ \Psi & & \\ h & \longmapsto & \lambda^* h \end{array}$

$$(\lambda^* h)(e') = h(\lambda(e')) = h(\Delta \cdot e') = (\Delta h)(e')$$

$\Rightarrow \lambda^* h = \Delta h \Rightarrow$ The diagram$^{(**)}$ is commutative. ⌟

Set $I_k' = \{f_1', \cdots f_k'\}$ and consider the following array of sequences of $\mathcal{O}$-modules, which are exact by the regular sequence property of the f_i':

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O} & \xrightarrow{f_1' \text{ multipli.}} & \mathcal{O} & \longrightarrow & \mathcal{O}/I_1' & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{O}/I_1' & \xrightarrow{f_2'} & \mathcal{O}/I_1' & \longrightarrow & \mathcal{O}/I_2' & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longrightarrow & \mathcal{O}/I_{r-1}' & \longrightarrow & \mathcal{O}/I_{r-1}' & \longrightarrow & \mathcal{O}/I_r' (= I') & \longrightarrow & 0 .
\end{array}$$