Summarizing, we have shown that the Fourier series mapping $C^0(T) \longrightarrow \mathcal{F}$ leads to inclusions

$$C^s(T) \subset H_s,$$
$$H_{s+[n/2]+1} \subset C^s(T),$$
$$C^\infty(T) = H_\infty.$$

A useful remark is that the proof of the Sobolev lemma gives an estimate

$$\sup_{x\in T} |D^\alpha \varphi(x)| \leq C_\alpha \|\varphi\|_{[n/2]+1+[\alpha]}.$$

Γ $\varphi \in H_{[n/2]+1+[\alpha]}$. $\varphi = \sum_\xi \varphi_\xi e^{i\langle \xi, x\rangle}$

$$|D^\alpha \varphi(x)| = \left|\sum \xi^\alpha \varphi_\xi e^{i\langle\xi,x\rangle}\right| \leq \left|\sum (1+\|\xi\|^2)^{\frac{[\alpha]}{2}} \varphi_\xi e^{i\langle\xi,x\rangle}\right|$$

$$\leq \left|\sum \frac{\left((1+\|\xi\|^2)^{[\frac{n}{2}]+1}(1+\|\xi\|^2)^{[\alpha]}|\varphi_\xi|^2\right)^{\frac{1}{2}}}{\left((1+\|\xi\|^2)^{[\frac{n}{2}]+1}\right)^{\frac{1}{2}}}\right|$$

$$\leq \left(\sum_\xi (1+\|\xi\|^2)^{[\frac{n}{2}]+1+[\alpha]}|\varphi_\xi|^2\right)^{\frac{1}{2}} \sum \frac{1}{\left((1+\|\xi\|^2)^{[\frac{n}{2}]+1}\right)^{\frac{1}{2}}}$$

$$= \|\varphi\|_{[\alpha]+[n/2]+1} \sum \left(\frac{1}{(1+\|\xi\|^2)^{[\frac{n}{2}]+1}}\right)^{\frac{1}{2}}$$

$\Rightarrow$ Since $\sum \left(\frac{1}{(1+\|\xi\|^2)^{[\frac{n}{2}]+1}}\right)^{\frac{1}{2}} < C$,

$|D^\alpha\varphi(x)| \leq C\|\varphi\|_{[\alpha]+[n/2]+1}$ for some constant C.

Rellich Lemma. For $s>r$, the inclusion $H_s \subset H_r$ is compact.