$\Rightarrow$ Contradiction, since $\overline{P_{23}} \in L_2 \cap L_3$ and $\overline{P_{23}}$ is a double point $\Rightarrow$ $L_1 \cap L_2 \cap L_3 \neq \overline{P_{23}}$. $\Rightarrow$ Similarly, we can show that $h \neq h_i$ for all i. ⌟

To do this, we recall that the points P_{ij}, P_{jk}, P_{ik}, q_1, q_2 and q_3 all lie on a conic curve, and hence so do their images $\overline{P_{ij}}$, $\overline{P_{jK}}$, $\overline{P_{iK}}$, $\overline{q_1}$, $\overline{q_2}$ and $\overline{q_3}$.

⌜ $h \cap \tilde{S}$ is a conic curve with multiplicity 2 where $h = \overline{P_i L_j}$. See P 774 ~ P775.
$P_0 \notin h \cap \tilde{S}$ $\Rightarrow$ $r = \pi_{p_0} : \tilde{S} \cap h \longrightarrow \mathbb{P}^2$, $\tilde{S} \subset h \subset \mathbb{P}^3$
$\Rightarrow$ By the result on P772, $\deg(\tilde{S} \cap h) = \deg(\pi_{p_0}(\tilde{S} \cap h)) = 2$. Here we consider $\tilde{S} \cap h$ as a set in $\mathbb{P}^3$. ⌟

In particular, this means that no three of these points are collinear, i.e., that $\overline{q_1}$, $\overline{q_2}$, and $\overline{q_3}$ must lie off the lines L_i, L_j and L_K.

⌜ Suppose three of the points are collinear.
$\Rightarrow$ Three of the p_i's are collinear. $\Rightarrow$ $r : \tilde{S} \to \mathbb{P}^2$ is not a 2-sheeted cover, in other words, $\exists$ a line l contained in $\tilde{S}$ s.t l contains three of the p_i's. Consider $\{\sigma(p) \cap X\}_{p \in l}$
$\Rightarrow$ It is a linear system on X as we saw $T_{x_0}(G) \cap$ where $x_0 = \varphi(l)$, $\varphi : G(2,4) \to \mathbb{P}^5$.