$I = (1, \cdots k)$, $J, K, L \subset (1, \cdots n)$ are index sets,
$z_J = z_{j_1} \cdots z_{j_q}$, $dz_J = dz_{j_1} \wedge \cdots \wedge dz_{j_q}$,
we may write

$$\varphi = \sum_{\substack{J \subset I \\ K \cap I = \phi}} \frac{\varphi_{JK}}{z_J} \frac{dz_{I-J}}{z_{I-J}} \wedge dz_K,$$

where φ_{JK} is holomorphic.

⌈ $f\varphi = \sum_{\# L = p} \psi_L \, dz_L$. where ψ_L is holomorphic.

$dz_L = dz_{I-J} \wedge dz_K$, $K \cap I = \phi$, and $J \subset I$.

$$z_1 \cdots z_k \varphi = \sum \psi_L \, dz_L = \sum_{\substack{J \subset I \\ K \cap I = \phi}} \psi_{I,J,K} \, dz_{I-J} \wedge dz_K$$

$$\Rightarrow \quad \varphi = \sum_{\substack{J \subset I \\ K \cap I = \phi}} \frac{\psi_{J,K}}{z_J} \frac{dz_{I-J}}{z_{I-J}} \wedge dz_K$$

⌋

Computing modulo terms T such that fT is holomorphic,

$$d\varphi \equiv - \sum_{J,K} \sum_{j \in J} \frac{\varphi_{JK}}{z_J} \frac{dz_j}{z_j} \wedge \frac{dz_{I-J}}{z_{I-J}} \wedge dz_K$$

$$= \sum_{L,K} \psi_{LK} \frac{dz_L}{z_L} \wedge dz_K,$$

where

$$z_{I-L} \psi_{LK} = \pm \sum_{i \in L} \frac{\varphi_{(I-L) \cup (i), K}}{z_i}$$

is holomorphic.