It remains to prove that $d\langle \tau, \tau' \rangle = \langle D^*\tau, \tau' \rangle + \langle \tau, D^*\tau' \rangle$.

$D^* e_i^* = \sum \theta_{ij}^* e_j^*$

$$d(e_i^*(e_k)) = D^* e_i^*(e_k) + e_i^*(De_k)$$

with $d(e_i^*(e_k)) = 0$, $D^*e_i^*(e_k) = \theta_{ik}^*$, $De_k = (\sum \theta_{kl} e_l)$, $e_i^*(De_k) = \theta_{ki}$:

$$0 = \theta_{ik}^* + \theta_{ki}$$

$\Rightarrow {}^t\theta^* = -\theta \Rightarrow \theta^*$ is skew-hermitian. $\left({}^t\bar{\theta} + \theta = 0,\ -\bar{\theta}^* + {}^t\theta^*\right)$

$$\Rightarrow d\langle e_i^*, e_j^* \rangle \overset{?}{=} \langle D^* e_i^*, e_j^* \rangle + \langle e_i^*, D^* e_j^* \rangle$$

$$0 = \theta_{ij}^* + \bar{\theta}_{ji}^* = 0$$

and $f e_i^*$, $g e_j^*$,

$d\langle f e_i^*, g e_j^* \rangle = d(f\bar{g})\delta_{ij}$

$\langle D^* f e_i^*, g e_j^* \rangle = \langle df \otimes e_i^* + f D e_i^*, g e_j^* \rangle = df\bar{g}\delta_{ij} + f\bar{g}\langle De_i^*, e_j^* \rangle$

$\langle f e_i^*, D^* g e_j^* \rangle = \langle f e_i^*, dg \otimes e_j^* + g D^* e_j^* \rangle$

$$= f\, d\bar{g}\, \delta_{ij} + f\bar{g} \langle e_i^*, D^* e_j^* \rangle$$

$$\Rightarrow d\langle f e_i^*, g e_j^* \rangle = \langle D^* f e_i^*, g e_j^* \rangle + \langle f e_i^*, D^* g e_j^* \rangle$$

$\Rightarrow D^*$ is compatible with the metric.

Return to the general discussion.
Given a connection D on $E \longrightarrow M$, we can define operators

$$D : \mathcal{A}^p(E) \longrightarrow \mathcal{A}^{p+1}(E).$$ by forcing Leibnitz's rule

$$D(\psi \otimes \xi) = d\psi \otimes \xi + (-1)^p \psi \wedge D\xi. \quad \text{for } \psi \in \mathcal{A}^p(U),\ \xi \in \mathcal{A}^0(E)(U).$$