Chapter 7 Problem 4. (C)

Claim: $\iota(v)(\omega_1 \wedge \omega_2) = \iota(v)(\omega_1) \wedge \omega_2 + (-1)^{\deg \omega_1} \omega_1 \wedge \iota(v)(\omega_2)$.

(See Spivak. Introduction to Diff. Geometry Vol I)

pf) First of all, we want to show that if $\{v_1 \cdots v_n\}$ is a unitary frame, with its dual frame $\{\varphi_1 \cdots \varphi_n\}$, then

$$\iota(v_j)(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}) = \begin{cases} 0, & j \neq \text{any } i_\alpha \\ (-1)^{\alpha-1} \varphi_{i_1} \wedge \cdots \wedge \hat{\varphi}_{i_\alpha} \wedge \cdots \wedge \varphi_{i_k} & \text{if } j = i_\alpha. \end{cases}$$

Clearly, if $j \neq$ any i_α, then it is 0.

If $j = i_\alpha$,

$$\iota(v_j)(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k})(u_1, u_2, \quad \hat{u}_\alpha \cdots u_k)$$

$$= \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_j, u_1, u_2, \cdots \hat{u}_\alpha \cdots u_k)$$

$$= \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(u_1, u_2, \cdots v_j, \cdots u_k) \quad (-1)^{\alpha-1}$$

$$= \sum_{\sigma \in S_{k-1}} \epsilon(\sigma) \; \varphi_{i_1}(u_{\sigma(1)}) \cdots \hat{\varphi}_{i_\alpha}(v_j) \cdots \varphi_{i_k}(u_{\sigma(k)})$$

$$= (-1)^{\alpha-1} \varphi_{i_1} \wedge \cdots \wedge \hat{\varphi}_{i_\alpha} \wedge \cdots \wedge \varphi_{i_k}.(u_1, u_2, \cdots \hat{u}_\alpha \cdots u_k)$$

$\Rightarrow$ We proved the desired.

Let $\omega_1 = \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_p} = \varphi_I$ & $\omega_2 = \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_q} = \varphi_J$. $v = v_j$.

① $I \cap J \neq \phi$ Say. $\varphi_{i_\alpha} = \varphi_{j_\beta}$

$\Rightarrow \omega_1 \wedge \omega_2 = 0$

Unless $v_j = v_{i_\alpha}$ RHS $= 0 \Rightarrow$ done.

If $v_j = v_{i_\alpha}$ $\quad \iota(v_{i_\alpha})(\varphi_{i_1} \wedge \cdots) = (-1)^{\alpha-1} \varphi_{i_1} \wedge \cdots \wedge \hat{\varphi}_{i_\alpha} \cdots \wedge \varphi_{i_p}$.

$\iota(v_{i_\alpha})(\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_\beta} \wedge \cdots) = (-1)^{\beta-1} \varphi_{j_1} \wedge \cdots \wedge \hat{\varphi}_{j_\beta} \cdots \wedge \varphi_{j_q}$.