It is proved in Real & Complex Analysis (by Rudin) (P97 2nd edition). It shows that $\{e^{i\langle\xi,x\rangle}\}$ is a complete set in $L^2(T)$ which implies. $\Sigma|\varphi_\xi|^2 = \int_T |\varphi|^2 dx$, (P90. Th 4.18 & P94 4.24)

By assuming P97, we can finish the argument as follows.

$$\int_T |\varphi|^2 = \int_T \Sigma \varphi_\xi e^{i\langle\xi,x\rangle} \Sigma \overline{\varphi_{\xi'}} e^{-i\langle\xi',x\rangle}$$

$$= \int_T \Sigma \varphi_\xi \overline{\varphi_{\xi'}} e^{i\langle\xi-\xi',x\rangle} dx. \quad \cdots ⊛$$

$$\left| \|\hat{\varphi}\|_2^2 - \|\hat{\varphi}_R\|_2^2 \right| < \epsilon^2 \quad \text{if } R \text{ is large enough.}$$

$$" (\|\hat{\varphi}\|_2 + \|\hat{\varphi}_R\|_2) | (\|\hat{\varphi}\|_2 - \|\hat{\varphi}_R\|_2) |$$

since $\|\hat{\varphi} - \hat{\varphi}_R\|_2 \geq \left| \|\hat{\varphi}\|_2 - \|\hat{\varphi}_R\|_2 \right|$ (by Cauchy-Schwartz inequality)

Here $\hat{\varphi} = \Sigma \varphi_\xi e^{i\langle\xi,x\rangle}$, $\hat{\varphi}_R = \sum_{|\xi|\leq R} \varphi_\xi e^{i\langle\xi,x\rangle}$.

$$\Rightarrow \int_T \Sigma \varphi_\xi \overline{\varphi_{\xi'}} e^{i\langle\xi-\xi',x\rangle} dx = \int_T \hat{\varphi}\,\overline{\hat{\varphi}} = \int_T |\hat{\varphi}|^2 = \|\hat{\varphi}\|_2^2$$

$$\Rightarrow ⊛ < \|\hat{\varphi}_R\|_2^2 + \epsilon^2. \Rightarrow$$

$$⊛ \leq \|\hat{\varphi}_R\|_2^2 + \epsilon^2 = \sum_R \int_T \varphi_\xi \overline{\varphi_{\xi'}} e^{i\langle\xi-\xi',x\rangle} dx + \epsilon^2$$

$$= \sum_R |\varphi_\xi|^2 + \epsilon^2 \leq \sum_\xi |\varphi_\xi|^2 + \epsilon^2.$$

$\Rightarrow$ Since ϵ is arbitrary, $\int_T \Sigma \varphi_\xi \overline{\varphi_{\xi'}} e^{i\langle\xi-\xi',x\rangle} dx \leq \sum_\xi |\varphi_\xi|^2$.

From $\left| \|\hat{\varphi}\|_2^2 - \|\hat{\varphi}_R\|_2^2 \right| < \epsilon^2$, i.e.

$$\left| \|\hat{\varphi}\|_2^2 - \sum_R |\varphi_\xi|^2 \right| < \epsilon^2, \quad \text{as } R \to \infty$$

$$\|\hat{\varphi}\|_2^2 = \lim_{R\to\infty} \sum_R |\varphi_\xi|^2 = \sum_\xi |\varphi_\xi|^2.$$