I think $f(W) = f(S \cap W)$ is not correct.

For example, $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ be a function defined by $f(z_1, z_2) = z_1 + z_2^2$.

$\Rightarrow \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right) = (1, 2z_2)$. Consider $\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right)\Big|_{\mathbb{C}\times 0}$

$= (1, 0)$. $\Rightarrow \mathrm{rank}(f_*|_{\mathbb{C}\times\{0\}}) = \mathrm{rank}(f_*)$.

But $\exists$ no bounded open set $W \ni (0,0)$. s.t $f(W) = f(W \cap \mathbb{C}\times\{0\})$, except $W = \mathbb{C}^2$, W

Wrong Again.

Since we can choose a coordinate chart $U \ni p_0$ s.t

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & \mathbb{C}^n \\ \cup & & \uparrow \\ U\cap S & \xrightarrow{\varphi|} & \mathbb{C}^k, \end{array}$$

we may assume that

Jacobian of $f : \mathbb{C}^n \longrightarrow \mathbb{C}^N$ has maximum rank k at 0.

and $f|\mathbb{C}^k \longrightarrow \mathbb{C}^N$ has rank k too.

Since $N \geq k$, we can divide into two cases.

(i) $N = k$

Consider $f|\mathbb{C}^k \longrightarrow \mathbb{C}^N$

$\Rightarrow$ At the origin, Jacobian of f is nonsingular.

$\Rightarrow$ By the inverse function theorem, $\exists$ open set V_1, V_2 s.t $f : V_1 \longrightarrow V_2$ is biholomorphic.