Thus $\{\rho_i\}$ looks something like a partition of unity for the covering $\{U_i\}$ of U^* and may be used as such for any ω having first-order poles on the D_i.
Indeed, given

$$\omega = \frac{g(z)\, dz_1 \wedge \cdots \wedge dz_n}{f_1(z) \cdots f_n(z)}, \quad g(z) \in \mathcal{O}(U),$$

we see that

$$\rho_i \omega = \frac{\bar{f_i}\, g}{\|f\|^2} \frac{dz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots \hat{f_i} \cdots f_n} \in A^{n,0}(U_{\{i\}^\circ}),$$

so we can set

$$\xi_{\{i\}^\circ} = \pm \rho_i \omega, \quad \omega_{\{i\}^\circ} = \pm \bar\partial \rho_i \omega.$$

Pf $\quad U_{\{i\}^\circ} = U - (D_1 + \cdots + \hat{D_i} + \cdots + D_n)$
On $U_{\{i\}^\circ}$ $\quad f_1 \cdots \hat{f_i} \cdots f_n$ is holomorphic. $\quad\lrcorner$

Proceeding to the next step, for $j \neq i$,

$$\rho_j \omega_{\{i\}^\circ} \in A^{n,1}(U_{\{i,j\}^\circ}),$$

so we can set

$$\xi_{\{i,j\}^\circ} = \pm(\rho_i \omega_{\{j\}^\circ} - \rho_j \omega_{\{i\}^\circ}),$$

$$\omega_{\{i,j\}^\circ} = \pm 2\, \bar\partial \rho_i \wedge \bar\partial \rho_j \wedge \omega.$$

Pf $\quad \omega_{\{i\}^\circ} = \pm \bar\partial \rho_i \omega \in A^{n,0}(U_{\{i\}^\circ})$
$\rho_j\, \omega_{\{i\}^\circ} = \pm \rho_j \bar\partial \rho_i \omega$ is defined on $U - (D_1 + \cdots + \hat{D_i} + \cdots + \hat{D_j} + \cdots + D_n) \Rightarrow \rho_j\, \omega_{\{i\}^\circ} \in A^{n,1}(U_{\{i,j\}^\circ})$. $\quad\lrcorner$

Continuing, we finally arrive at

$$\eta_\omega = \omega_{\{i\}} = n!\, (-1)^{i-1}\, \bar\partial \rho_1 \wedge \cdots \wedge \widehat{\bar\partial \rho_i} \wedge \cdots \wedge \bar\partial \rho_n \wedge \omega$$