$$\begin{array}{ccc} C_K(M.\mathbb{Z}) & \longrightarrow & H_{n-k}(M^{n-k}, M^{n-k-1}) \\ \Downarrow & & \Downarrow \\ r = \sum a_\alpha \sigma_\alpha^k & \longmapsto & \sum a_\alpha \Delta_\alpha^{n-k} \end{array}$$

$$\downarrow \cong$$

$$\mathrm{Hom}(H_{n-k}(M^{n-k}, M^{n-k-1}), \mathbb{Z})$$

$$\Downarrow$$

$$\sum a_\alpha \tilde{\Delta}_\alpha^{n-k} \quad \text{where} \quad \tilde{\Delta}_\alpha^{n-k}(\Delta_\beta^{n-k}) = \delta_{\alpha\beta}$$

$$\| $$

$$\#(\sigma_\alpha^k \cdot \Delta_\beta^{n-k})$$

$$\Rightarrow \quad Dr(\lambda) = \sum a_\alpha \tilde{\Delta}_\alpha^{n-k}(\lambda) = \#(\sum a_\alpha \sigma_\alpha^k \cdot \lambda) = \#(r \cdot \lambda).$$

Q.E.D

Note that. if r is a torsion class, $\#(r.\lambda)=0$ for all $\lambda \in H_{n-k}(M.\mathbb{Z})$.

$$H^{n-k}(M.\mathbb{Z}) \supset \mathrm{Hom}(H_{n-k}(M.\mathbb{Z}).\mathbb{Z}).$$

$\Rightarrow$ Dr goes to a torsion part of $H^{n-k}(M.\mathbb{Z})$, not into $\mathrm{Hom}(H_{n-k}(M.\mathbb{Z}).\mathbb{Z})$.

A somewhat weaker version of Poincaré duality is the statement that the map

$$H_k(M.\mathbb{Q}) \xrightarrow{P} H_{n-k}(M.\mathbb{Q})^* \cong H^{n-k}(M.\mathbb{Q})$$

given by $P(A)(B) = \#(A \cdot B)$ is an isomorphism, omitting the fact that the intersection pairing is unimodular.

Via the de Rham isomorphism.

$$H_{DR}^{n-k}(M) \longrightarrow H^{n-k}(M.\mathbb{C})$$