Setting $R' = R \cap \mathcal{O}^{(k-1)}$, in the exact sequence

$$0 \longrightarrow R' \longrightarrow R \longrightarrow R/R' \longrightarrow 0$$

both R' and $R/R' \subset \mathcal{O}$ are finitely generated, and hence so is R.

$\ulcorner$ $$0 \longrightarrow R' = R \cap \mathcal{O}^{(k-1)} \longrightarrow \mathcal{O}^{(k-1)} \xrightarrow{\pi} \underset{\{g_1 m_1 + \cdots + g_{k-1} m_{k-1}\}}{\overset{M'}{\shortparallel}} \longrightarrow 0$$

$\Rightarrow$ By the induction assumption, R' is finitely generated.

$$R'' = \{g \in \mathcal{O} : g m_k \in M'\} \xrightarrow{\phi} R/R'$$
$$g \longmapsto (0, \cdots 0, g) + R' = (\{(g_1, \cdots g_{k-1}, 0) \mid g_1 m_1 + \cdots + g_{k-1} m_{k-1} = 0\})$$

$\Rightarrow$ Clearly ϕ is onto. $\phi(g) = R' \Rightarrow (0, \cdots 0, g) \in R'$
$\Rightarrow g = 0 \Rightarrow \phi$ is one to one. $\Rightarrow \exists\ \psi : R/R' \longrightarrow R''$ defined by $(0, \cdots 0, g) + R' \longmapsto g$.
$\Rightarrow R'' \cong R/R' \Rightarrow R/R'$ is finitely generated, since $R'' \subset \mathcal{O}$ is finitely generated.
$\Rightarrow$ From the exact sequence

$$0 \longrightarrow R' \longrightarrow R \longrightarrow R/R' \longrightarrow 0,$$

given any $g \in R$, $g + R' = a_1(f_1 + R') + \cdots + a_k(f_k + R')$ where $\{f_1 + R', \cdots f_k + R'\} = R/R'$. $\Rightarrow g - a_1 f_1 - \cdots - a_k f_k \in R' \Rightarrow g - a_1 f_1 - \cdots - a_k f_k = b_1 h_1 + \cdots + b_\ell h_\ell$, where $R' = \{h_1, \cdots h_\ell\} \Rightarrow R$ is finitely generated. $\lrcorner$

As an example of $\mathcal{O}$-modules, in addition to the free $\mathcal{O}$-modules mentioned above, the most important ones are