3. Chern Classes

Definitions.

In this section we will give the definition and some properties of the basic topological invariants of complex vector bundles, the Chern classes. We will not be concerned with holomorphic bundles until later: for the present, all our manifolds and vector bundles will be simply C^∞.

We begin by recalling some of the definitions of Section 5 of Chapter 0. Let M be a manifold, $E \xrightarrow{\pi} M$ a complex vector bundle, and $\mathcal{A}^p(E)$ the sheaf of E-valued p-forms, that is, the sheaf of C^∞ sections of the bundle $\Lambda^p T^*(M) \otimes E$. We define a connection D on E to be an operator

$$D: \mathcal{A}^0(E) \longrightarrow \mathcal{A}^1(E)$$

satisfying Leibnitz' rule

$$D(f \cdot \xi) = df \otimes \xi + f \cdot D\xi$$

for $f \in C^\infty(U)$, $\xi \in \mathcal{A}^0(E)(U)$. If $\varphi_\alpha : E|_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{C}^n$ is a trivialization of E over $U_\alpha \subset M$, then we can identify sections ξ of E over U_α with n-vectors $\xi_\alpha = {}^t(\xi_{\alpha,1}, \cdots \xi_{\alpha,n})$ of functions on U_α. If $\{e_{\alpha,i}\}$ is the frame for E over U_α given by the constant vectors $(0, \cdots 1, \cdots 0)$, we can write

$$De_{\alpha,i} = \sum \theta_{\alpha_{ij}} \otimes e_{\alpha_j}.$$

The matrix $\theta_\alpha = (\theta_{\alpha_{ij}})$ of 1-forms is called the connection matrix for D; we have for a general section $\xi \in \mathcal{A}^0(E)(U_\alpha)$

$$D\xi_\alpha = d\xi_\alpha + {}^t\theta_\alpha \cdot \xi_\alpha.$$