$$(p_1, p_2) \in \sigma\times\tau \cap \mu\times\nu \iff p_1 \in \sigma\cap\mu, \ \& \ p_2 \in \tau\cap\nu.$$

orientation for M

$$\#(\sigma\cdot\ ) [v_{11} \cdots v_{1n}] = [\sigma_1 \cdots \sigma_k, \mu_1 \cdots \mu_{n-k}]$$
$$\#(\tau\cdot\nu) [v_{21} \cdots v_{2n}] = [\tau_1 \cdots \tau_{n-k}, \nu_1 \cdots \nu_k]$$

$$[\sigma_1 \cdots \sigma_k \ \mu_1 \cdots \mu_{n-k}, \tau_1 \cdots \tau_{n-k} \cdots \nu_1 \cdots \nu_k] \ \#(\sigma\cdot\mu)\cdot\#(\tau\cdot\nu)$$
$= [v_{11} \cdots v_{1n}][v_{21} \cdots v_{2n}] = [v_{11} \cdots v_{1n}. v_{21} \cdots v_{2n}]$ orientation for $M\times M$.

$\Rightarrow$ Since $[\sigma_1 \cdots \sigma_k, \mu_1, \cdots \mu_{n-k} \ \tau_1 \cdots \tau_{n-k} \cdots \nu_1 \cdots \nu_k]$
$= [\sigma_1, \cdots \sigma_k. \tau_1 \cdots \tau_{n-k}, \mu_1 \cdots \mu_{n-k}. \nu_1 \cdots \nu_k] (-1)^{n-k} = \#(\sigma\times\tau, \mu\times\nu)$
$[v_{11} \cdots v_{2n}] (-1)^{n-k}$,

$$\#(\sigma\times\tau, \mu\times\nu)(-1)^{n-k} = \#(\sigma\cdot\mu).\#(\tau\cdot\nu) \ //$$

Note that this formula holds for any $(n-k')$-cycle μ & k-cycle ν; if $k \neq k'$, both sides zero.

By Künneth, such products of cycles generate $H_n(M\times M. \mathbb{Q})$, and so it follows that the form $\pi_1^*\varphi \wedge \pi_2^*\psi$ is Poincare dual to the cycle $(-1)^{n-k}\sigma\times\tau$, i.e for any n-cycle η in $M\times M$,

$$(-1)^{n-k} \int_\eta \pi_1^*\varphi \wedge \pi_2^*\psi = \#(\sigma\times\tau\cdot\eta)$$

We apply this in particular to the diagonal $\Delta \subset M\times M$. On the other hand,

$$\int_\Delta \pi_1^*\varphi \wedge \pi_2^*\psi = \int_M \Delta^*(\pi_1^*\varphi\wedge\pi_2^*\psi) = \int_M (\pi_1\circ\Delta)^*\varphi \wedge (\pi_2\circ\Delta)^*\psi$$

$$= \int_M \varphi\wedge\psi \quad \text{since} \quad \pi_1\circ\Delta = \pi_2\circ\Delta = \text{identity}.$$