and so

$$h^0(K-p-q) < h^0(K-p) \iff h^0(p+q) = 1.$$

Γ $h^0(K-p-q) = h^0(p+q) + g - \deg(p+q) - 1 = h^0(p+q) + g - 3$

$\Rightarrow$ $h^0(K-p-q) < h^0(K-p) \iff h^0(p+q) + g - 3 < g - 1$

$\iff h^0(p+q) < 2 \iff h^0(p+q) = 1$ or 0.

$h^0(p+q) = \dim H^0(S, \mathcal{O}(p+q)) \Rightarrow \exists\ s$ section of $[p+q]$ s.t $(s=0) = p+q$. $\Rightarrow$ s is holomorphic section of $[p+q]$

$\Rightarrow h^0(p+q) > 0 \Rightarrow h^0(p+q) = 1$ ⌟

Thus ι_K fails to be an embedding if and only if there exists a meromorphic function on S having only two poles, i.e, if S can be expressed as a two-sheeted branched covering of $\mathbb{P}^1$. Such a Riemann surface is called hyperelliptic.

Γ($\Rightarrow$) ι_K fails to be an embedding $\Rightarrow$ $h^0(p+q) \neq 1$.

$\Rightarrow H^0(S, \mathcal{O}(p+q)) \ni \sigma$, $(\sigma=0) \neq p+q$. $\Rightarrow$ $\frac{\sigma}{s_0}$ is a meromorphic function on S $\Rightarrow$ There are two cases:

(1) $(\sigma=0) = p'+q'$ $\quad p=p'$ $\quad q \neq q'$

$\Rightarrow$ σ/s_0 has only one pole at q $\Rightarrow$ Impossible since $g(S) \geq 2$ see p222

(2) $(\sigma=0) = p'+q'$, $\quad p \neq p'$, $\quad q \neq q'$.

$\Rightarrow$ σ/s_0 has two poles.

($\Leftarrow$) Let f be a meromorphic function on S having only two poles. $\Rightarrow$ $f \cdot s_0 \in H^0(S, \mathcal{O}(p+q))$.

If $(f=0) = -p-q+p'+q'$ $f \cdot s_0$ is a meromorphic section.