The Bertini theorem is a refinement of Sard's theorem for this mapping.

Γ $p \in M-B$. Suppose $p \in D_{\lambda_1} \cap D_{\lambda_2}$.

$\Rightarrow$ Consider $\Delta \ni p$.

$$D_{\lambda_1} = (f(z) + \lambda_1 g(z) = 0)$$

$$D_{\lambda_2} = (f(z) + \lambda_2 g(z) = 0)$$

$$f(p) + \lambda_1 g(p) = 0. \qquad f(p) + \lambda_2 g(p) = 0.$$

Consider $\frac{f}{g}$. $\Rightarrow$ f/g is a global meromorphic function on M. $\Rightarrow$ $\frac{f(p)}{g(p)} = -\lambda_1 \ \& \ -\lambda_2$. $\Rightarrow$ $\lambda_1 = \lambda_2$

$\Rightarrow$ $p \in D_\lambda$ where $\lambda = \lambda_1 = \lambda_2$. $\Rightarrow$ This proves the well-definedness.

Given $p \in M-B$, since $f(p) \neq 0$ & $g(p) \neq 0$,
let $\lambda = -\frac{f(p)}{g(p)}$. $\Rightarrow$ $p \in D_\lambda$.

$$\mathbb{P}(H^0(M, \mathcal{O}([D]))) = \mathbb{P}^1.$$

$\Rightarrow$ $\exists$ two linearly independent global holomorphic sections $s_1, s_2 \in H^0(M, \mathcal{O}([D]))$. Let $D_{\lambda_i} = (s_i)$.

Define $\frac{s_1}{s_2}$ which is a meromorphic function on M.

If $p \in B = \bigcap_{i=1}^{2} D_{\lambda_i}$, $s_1(p) = s_2(p) = 0$, $\Rightarrow$ We do not know what is $\frac{s_1(p)}{s_2(p)}$.

If $p \in M-B$, $\exists$ a polydisc $\Delta \ni p$ s.t.

$$s_1|_\Delta + \lambda_1 s_2|_\Delta = 0 \quad \text{or} \quad \lambda_2 s_1|_\Delta + s_2|_\Delta = 0 \quad \text{at } p.$$

If $s_1|_\Delta(p) = 0$, $\lambda_1 = 0$, if $s_2|_\Delta(p) = 0$ $\lambda_2 = 0$. $\Rightarrow$ $\frac{s_1}{s_2}(p) = 0 \ \& \ \infty$.