(2) k is nonnegative

$$\mathcal{O}_{\mathbb{P}^n}[kH] = \mathcal{O}([kH] \otimes K_{\mathbb{P}^n} \otimes K_{\mathbb{P}^n}^*) = \Omega^n([kH] \otimes K_{\mathbb{P}}^*)$$

$$= \Omega^n([kH] \otimes [(n+1)H]) \quad \text{(by p150)}.$$
$$= \Omega^n([(k+n+1)H]) = \Omega^n((k+n+1)H)$$

$$\Rightarrow \quad H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) = H^q(\mathbb{P}^n, \Omega^n_{\mathbb{P}^n}((k+n+1)H))$$

$\Rightarrow$ Since $[(k+n+1)H]$ is positive line bundle, by Kodaira - Nakano theorem, in case $q+n > n$,

$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kH)) = 0$. //

The Lefschetz Theorem on Hyperplane Sections

Using the Kodaira vanishing theorem, we can give a proof of the famous Lefschetz theorem relating the homology of a projective variety to that of its hyperplane sections.

Let M be an n-dimensional compact complex manifold and $V \subset M$ a smooth hypersurface with $L = [V]$ positive - e.g. $M \subset \mathbb{P}^N$ a submanifold of projective space and $V = M \cap V$ a hyperplane section of M. Then we have the
(See p150)

Lefschetz Hyperplane Theorem. The map

$$H^q(M, \mathbb{Q}) \longrightarrow H^q(V, \mathbb{Q})$$

indued by the inclusion $i : V \hookrightarrow M$ is an isomorphism for $q \leq n-2$ and injective for $q = n-1$.

pf). It will suffice to prove the result over $\mathbb{C}$. By the Hodge decomposition