Choose three noncollinear points; call them P_1, P_2, and P_3 and let C be a conic containing the remaining points $P_4, P_5, \cdots P_8$. By hypothesis, the cubics

$$C + L_{12}, \quad C + L_{13}, \quad \text{and} \quad C + L_{23}$$

each contain all eight points; since each of the points P_1, P_2 and P_3 lies outside one of the lines L_{12}, L_{13}, and L_{23}, it follows that the conic C contains all eight. Q.E.D.

By the hypothesis that any cubic passing through any seven of the points contains them all, since $C + L_{12}$, $C + L_{13}$ and $C + L_{23}$ contain 7 points, they contain all 8 points. Since P_1, P_2, P_3 are not collinear, $P_1 \notin L_{23}$. $\Rightarrow$ $P_1 \in C + L_{23} \Rightarrow P_1 \in C$. Similarly, $P_2 \in L_{13} + C \Rightarrow P_2 \notin L_{13} \Rightarrow P_2 \in C \Rightarrow$ Again $P_3 \in C$.
$\Rightarrow P_1, P_2, P_3, P_4 \cdots P_8 \in C \Rightarrow C$ contains all eight.
Since $H^0(\mathbb{P}^2, \mathcal{O}(2H)) = \langle \tau_1, \tau_2, \cdots \tau_6 \rangle$, $\exists\ b_1, b_2 \cdots b_6$ not all zero s.t $b_1\tau_1 + \cdots + \tau_6 = 0$ at $P_4, P_5, \cdots P_8$. Point is $\#\{P_4, \cdots P_8\} = 5 < 6$.

$$\left.\begin{array}{l} b_1\tau_1(P_4) + \cdots + b_6\tau_6(P_4) = 0 \\ \quad\vdots \qquad\qquad\qquad\quad \vdots \\ b_1\tau_1(P_8) + \cdots + b_6\tau_6(P_8) = 0 \end{array}\right\} \text{has nontrival solutions.}$$

The statement of the lemma also holds in case P_1 is infinitely near P_2, that is, if P_1 is a point on the exceptional divisor E of the blow-up $\widetilde{\mathbb{P}^2} \xrightarrow{\pi} \mathbb{P}^2$