$$\begin{cases} G = 0 & \text{on } \mathcal{H}^{p,q}(M). \\ G\varphi = \frac{1}{\lambda_m}\varphi, & \varphi \in E\left(\frac{1}{1+\lambda_m}\right) = E(\rho_m). \end{cases},$$

then G is a compact, self-adjoint operator with spectral decomposition

$$\mathcal{H}_0^{p,q}(M) = \mathcal{H}^{p,q}(M) \oplus \left(\bigoplus_m E(\rho_m)\right)$$

where $G\varphi = \frac{\rho_m}{1-\rho_m}\varphi, \quad \varphi \in E(\rho_m)$.

$\boxed{\Gamma}$ $\varphi = \varphi_1 + \varphi_2 + \cdots \in V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n} \oplus \cdots$

$$\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2$$
$$= \lambda_1\varphi_1 + \lambda_2\varphi_2 + \cdots$$

$$\underset{\|\Delta\varphi\|_0^2}{\langle \Delta\varphi, \Delta\varphi \rangle} = \lambda_1^2\|\varphi_1\|_0 + \lambda_2^2\|\varphi_2\|^2 + \cdots$$
$$\geq \lambda_1^2(\langle\varphi, \varphi\rangle) = \lambda_1^2\|\varphi\|_0^2$$

$$\Rightarrow \quad \|\Delta\varphi\|_0 \geq \lambda_1\|\varphi\|_0$$

$\langle G\varphi, \varphi\rangle \geq 0 \Rightarrow G$ is self-adjoint.

To show G is compact, $\varphi \in \mathcal{H}_0^{p,q}$, we need to show $G(\varphi) \in \mathcal{H}_1^{p,q}$.

$\Rightarrow \varphi = \oplus \varphi_m \qquad \varphi_m \in E\left(\frac{1}{1+\lambda_m}\right) = E(\rho_m)$

$G(\varphi) = \oplus G(\varphi_m) = \oplus \frac{1}{\lambda_m}\varphi_m.$

equivalent to $\| \ \|_1$.

$$\langle G(\varphi), (I+\Delta)G(\varphi)\rangle = \mathcal{D}(G(\varphi)) = \langle \oplus\tfrac{1}{\lambda_m}\varphi_m, \oplus\tfrac{1}{\lambda_m}\varphi_m + \oplus\varphi_m\rangle$$
$$= \sum \frac{\|\varphi_m\|_0^2}{\lambda_m^2} + \sum \frac{\|\varphi_m\|_0^2}{\lambda_m} = \sum_m \left(\frac{1}{\lambda_m^2} + \frac{1}{\lambda_m}\right)\|\varphi_m\|_0^2$$

$< \sum_m \|\varphi_m\|_0^2$ for large enough m since λ_m is greater than 1 ($\because \lambda_m \to \infty$ as $m \to \infty$).