$$ds^2 = \sum (\delta_{ij} + g_{ij})\, dz_i \otimes d\bar{z}_j,$$

where g_{ij} vanishes up to order k at z_0; we usually write

$$ds^2 = \sum (\delta_{ij} + [k])\, dz_i \otimes d\bar{z}_j .$$

Lemma. ds^2 is Kähler $\Leftrightarrow$ it osculates to order 1 to the Euclidean metric everywhere.

pf). ($\Leftarrow$) If $\omega = \frac{i}{2} \sum (\delta_{ij} + [\ ])\, dz_i \wedge d\bar{z}_j$

in some coordinate system around z_0, then $d\omega(z_0) = 0$

⌈ $d(\delta_{ij} + [1]) = 0 + d[1] = 0$ ⌋

($\Rightarrow$) Conversely, we can always find coordinates (z) for which $h_{ij}(z_0) = \delta_{ij}$; i.e

$$\omega = \frac{i}{2} \sum_{i,j,k} (\delta_{ij} + a_{ijk} z_k + a_{ij\bar{k}} \bar{z}_k + [1])\, dz_i \wedge d\bar{z}_j;$$

note that $h_{ij} = \overline{h_{ji}} \Rightarrow a_{ji\bar{k}} = \overline{a_{ijk}}$

and $d\omega = 0 = 0 \Rightarrow a_{ijk} = a_{kji}$.

⌈ z_0 goes to the origin in $\mathbb{C}^n$.

Suppose we have a coordinate chart (z), s.t $h_{ij}(z_0) \neq \delta_{ij}$

$\Rightarrow$ Consider

$$\begin{aligned} \omega_1 &= a_{11} z_1 + \cdots + a_{n1} z_n \\ \omega_2 &= a_{12} z_1 + \cdots + a_{n2} z_n \\ &\vdots \\ \omega_n &= a_{1n} z_1 + \cdots + a_{nn} z_n . \end{aligned}$$

Choose (a_{ij}) so that $\langle \frac{\partial}{\partial \omega_i}, \frac{\partial}{\partial \omega_j} \rangle_0 = \delta_{ij}$, i.e

since $\frac{\partial}{\partial \omega_i} = \sum b_{ki} \frac{\partial}{\partial z_k} = \sum \frac{\partial z_k}{\partial \omega_i} \frac{\partial}{\partial z_k}$.