for all $\eta \in A^{p,q}(M)$. Then $\psi \in \mathcal{H}^{p,q}_{s+2}(M)$.

For example, suppose that $\varphi \in \mathcal{H}^{p,q}_0(M)$ is an eigenfunction for the Laplacian, meaning that, for a constant λ, the equation $\Delta\varphi = \lambda\varphi$ holds in the weak sense. Then by the regularity lemma, $\varphi \in \mathcal{H}^{p,q}_s(M)$ for all s, and by the global Sobolev lemma we conclude that any eigenfunction for Δ is smooth.

We note that any eigenvalue $\lambda \geq 0$ ⇒ clear. from $\Delta \geq 0$., and $\lambda = 0 \Leftrightarrow$ φ is harmonic in the weak sense. By the regularity and Sobolev lemmas any such weakly harmonic form is C^∞ and harmonic in the usual sense.

We shall assume the Garding inequality and regularity lemma and go ahead and complete the proof of the Hodge theorem. After this done we shall prove the Garding inequality. The regularity lemma will be proved when we discuss smoothing of distribution in general. The reader who wishes to have the complete argument at hand may find the proof at the end of the subsection entitled "Smoothing and Regularity" in Section 1 of Chapter 3.

The basic Hilbert-space tool is the spectral theorem for compact self-adjoint operators, together with the principle of representing bounded linear functions by taking the inner product with a fixed vector, in the form of the following.

Lemma. Given $\varphi \in \mathcal{H}^{p,q}_0(M)$, $\exists$ a unique $\psi \in \mathcal{H}^{p,q}_1(M)$ s.t $(\varphi, \eta) = \mathcal{D}(\psi, \eta) = (\psi, (I+\Delta)\eta)$