We return now to our complex vector bundle $E \xrightarrow{\pi} M$ of rank n. Let $\{U_\alpha\}$ be an open cover of M with φ_α a trivialization of E over U_α and θ_α and Θ_α the connection and curvature matrices of the connection D on E in terms of φ_α. Then, since the wedge product is commutative on forms of even degree, for any invariant polynomial P of degree k on M_n the expression

$$P(\Theta_\alpha)$$

is a well-defined form of degree $2k$ on U_α; since

$$\Theta_\alpha = g_{\alpha\beta}\,\Theta_\beta \cdot g_{\alpha\beta}^{-1},$$

we see that

$$P(\Theta_\alpha) = P(\Theta_\beta)$$

in $U_\alpha \cap U_\beta$, so that $P(\Theta) = P(\Theta_\alpha)$ is a well-defined global $2k$-form on M, independent of the trivializations chosen.

Pf

$$P(\Theta_\alpha) = P(\Theta_{\alpha 11}, \Theta_{\alpha 12} \cdots \Theta_{\alpha 1n}, \Theta_{\alpha 21} \cdots \Theta_{\alpha 2n}, \cdots)$$

$$= P\begin{pmatrix} \Theta_{\alpha 11} & \cdots & \Theta_{\alpha 1n} \\ \vdots & & \\ \Theta_{\alpha n1} & \cdots & \Theta_{\alpha nn} \end{pmatrix}$$

$\Rightarrow$ Since P is homogeneous of deg k, $P(\Theta_\alpha)$ is $2k$-form.

If we choose a different trivialization φ_γ, we have

$$\Theta_\alpha = g_{\alpha\gamma}\,\Theta_\gamma \cdot g_{\alpha\gamma}^{-1}.$$

$\Rightarrow \quad P(\Theta_\alpha) = P(\Theta_\gamma).$ □

The basic fact is

Lemma. For P any invariant polynomial of degree k,