A note: this lemma will reappear as a consequence of the general duality theory discussed in Section 4 of Chapter 5. ☞ See P 715 ⌋

Let us return now to the blow-up $\pi: \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ of $\mathbb{P}^2$ at six points $p_1 \cdots p_6$ as specified earlier, and the linear system $|\widetilde{C}| = |\pi^* 3H - E_1 - E_2 - \cdots E_6|$. As an immediate consequence of the lemma, we see that the points $p_1, \cdots p_6$ impose independent conditions on cubics, so that $\dim |\widetilde{C}| = 3$.

☞ By the lemma, $p_1, p_2, \cdots p_6$ impose linearly independent conditions on cubics. $\Rightarrow \dim |\widetilde{C}| = \dim H^0(\mathbb{P}^2, \mathcal{J}_{p_1 \cdots p_6}(3)) - 1$ $= \dim |\mathcal{J}_{p_1 \cdots p_6}(3)| = \dim H^0(\mathbb{P}^2, \mathcal{O}(3)) - 6 - 1 = {}_5C_3 - 7 =$ $\frac{5 \cdot 4 \cdot 3}{6} - 7 = 10 - 7 = 3.$ ⌋

The remaining assertions 3a–d and 4a and b likewise follow from the lemma: respectively, they may be restated as saying that the points $p_1, p_2, \cdots p_6$, $p$ and $q$ impose independent conditions on cubics in case

3a. $p \neq q \in \mathbb{P}^2 - \{p_1, \cdots p_6\}$,

3b. $p$ infinitely near $p_i$, $q \in \mathbb{P}^2 - \{p_1, \cdots p_6\}$,

3c. $p$ infinitely near $p_i$, $q$ infinitely near $p_j$,

3d. $p \neq q$ infinitely near $p_i$,

4a. $q$ infinitely near $p \in \mathbb{P}^2 - \{p_1, \cdots p_6\}$,