$$c_1(\mathcal{E}^*) = c_1(\mathcal{L} \oplus \mathcal{O}) = c_1(\mathcal{L})\, c_0(\mathcal{O}) + c_0(\mathcal{L}) \cdot c_1(\mathcal{O})$$
$$= c_1(\mathcal{L}) \cdot 1 + c_0(\mathcal{L}) \cdot 0$$
$$= c_1(\mathcal{L}) \quad \text{in } H^2(S^*, \mathbb{Z}).$$

By excision theorem,

$$(S, S^*) \longleftrightarrow (\underbrace{S - (S - \amalg B_p)}_{\amalg B_p}, \underbrace{S^* - (S - \amalg B_p)}_{\amalg B_p^*})$$

induces an isomorphism on cohomology groups

$$H^2(\amalg B_p, \amalg B_p^*; \mathbb{Z}) \cong H^2(S, S^*; \mathbb{Z})$$
$$= \sum_{p \in Z} H^2(B_p, B_p^*; \mathbb{Z}) = 0, \quad \text{where } B_p^* = B_p - \{p\}.$$

since $H^2(B_p, B_p^*; \mathbb{Z}) = H^2(S^4, \mathbb{Z}) = 0$. ⌟

Actually this makes sense, since, very roughly speaking, giving $\mathcal{E}$ is the same as giving its Chern classes $c_1(E)$ and $c_2(E)$, and $c_2(E)$ is just Z. The assertion $c_1(\mathcal{L}) = -c_1(\mathcal{E})$ may be refined to

$$\mathcal{L} = \Lambda^2 \mathcal{E}^* \quad \text{in } \operatorname{Pic}(S) = H^1(S, \mathcal{O}^*),$$

which follows from the Levi Extension Theorem given in Section 2 of Chapter 3.

⌜ From $0 \longrightarrow \mathcal{L}|_{S^*} \longrightarrow \mathcal{E}^*|_{S^*} \longrightarrow \mathcal{O}_{S^*} \to 0$,

we get

$$\mathcal{E}^*|_{S^*} = \mathcal{L}|_{S^*} \oplus \mathcal{O}_{S^*}.$$

Suppose $\{g_{\alpha\beta}\}$ is a set of transition functions for