between $D_0 = \tilde{D}$ and $D_1 = D$. D_t has connection matrix $\theta_t = \tilde{\theta} + t\eta$, hence curvature matrix

$$\Theta_t = d(\tilde{\theta}+t\eta) - (\tilde{\theta}+t\eta)\wedge(\tilde{\theta}+t\eta).$$

Let P be an invariant polynomial of degree k. We claim that

$$[P(\Theta)] = [P(\tilde{\Theta})] \in H^{2k}_{DR}(M).$$

To prove this, we will consider the arc in $A^{2k}(M)$ given by

$$t \longmapsto P(\Theta_t)$$

and show that its tangent vector $(\partial/\partial t)\, P(\Theta_t)$ lies in the subspace $dA^{2k-1}(M) \subset A^{2k}(M)$; this will show that the image curve $t \longmapsto [P(\Theta_t)] \in H^{2k}_{DR}(M)$ is constant.

$\ulcorner$ $P(\Theta_t) - P(\Theta_s) = (t-s)\dfrac{\partial P(\Theta_l)}{\partial l}(c)$, where

$(c-s)(c-t) \leq 0$. by the mean value theorem.

$\Rightarrow$ $P(\Theta_t) - P(\Theta_s) \in dA^{2k-1}(M)$.

$\Rightarrow$ $[P(\Theta_t)] = [P(\Theta_s)]$.

Note that since $\Theta_t = d(\tilde{\theta}+t\eta) - (\tilde{\theta}+t\eta)\wedge(\tilde{\theta}+t\eta)$ and P is polynomial, we can apply the mean value theorem to our case without any difficulty. $\lrcorner$

The calculation goes as follows:

$$\frac{\partial}{\partial t}\Theta_t = d\eta - (\tilde{\theta}\wedge\eta + \eta\wedge\tilde{\theta}) - 2t\,\eta\wedge\eta,$$

hence