⊢ $\dim \Lambda^k V^* = {}_nC_k$. $\dim(\Lambda^n V^* \otimes \Lambda^{n-k} V) = \dim(\Lambda^n V^*)\dim(\Lambda^{n-k} V) = 1 \cdot {}_nC_{n-k}$.

Note that $i(e_1^*)(e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}) = e_{i_2} \wedge \cdots \wedge e_{i_m}$.

$$\begin{tikzcd}
\Lambda^k V^* \arrow[r] \arrow[d, "v^*\wedge"] & \Lambda^n V^* \otimes \Lambda^{n-k} V \arrow[d] \\
\Lambda^{k+1} V^* \arrow[r] & \Lambda^n V^* \otimes \Lambda^{n-(k+1)} V
\end{tikzcd}$$

$e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \longmapsto e_1^* \wedge \cdots \wedge e_n^* \otimes e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$, $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{n-k}$

$v^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* \qquad e_1^* \wedge \cdots \wedge e_n^* \otimes i(v^*)(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}})$

If $v = e_\ell$, ① $\ell \in \{i_1 \cdots i_k\}$. $v^* \wedge \cdots = 0$ &
$i(v^*)(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) = 0$

② $\ell \notin \{i_1 \cdots i_k\}$ and $i_{\ell-1} < \ell = j_p < i_\ell$

$\Rightarrow v^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = e_{i_\ell}^* \wedge \cdots \wedge e_{i_k}^* = (-1)^{\ell-1} e_{i_1} \wedge \cdots \wedge e_{i_k}^*$

$i(e_{i_\ell}^*)(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) = i(e_{j_p}^*)(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}})$

$= i(e_{j_p}^*)(e_{j_p} \wedge e_{j_1} \wedge \cdots \wedge \hat{e}_{j_p} \wedge \cdots e_{j_{n-k}})\ (-1)^{p-1}$

$= (-1)^{p-1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_p} \wedge \cdots \wedge e_{j_{n-k}}$

It is commutative up to sign. ⊥

Now to construct the Koszul complex let $e_1, \cdots e_r$ be the standard basis for $\mathbb{C}^r$, and set

$$\begin{cases} E_k = \mathcal{O} \otimes_{\mathbb{C}} \Lambda^k \mathbb{C}^r, \\ e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}, \quad J = (j_1, \cdots j_k) \subset (1, \cdots r). \end{cases}$$

Then E_k is a free $\mathcal{O}$-module with basis $\{e_J\}$, and we define

$$E_k \xrightarrow{\partial} E_{k-1}$$