Thus

$$v(x) = \sum_{j \neq i} (\alpha_j - \alpha_i)\, x_j \frac{\partial}{\partial x_j},$$

i.e., the matrix A_{p_i} for v near p_i is just the diagonal matrix with entries $(\alpha_j - \alpha_i)$, $j \neq i$.

$\ulcorner$

$$v(x) = \pi_* \sum_{i=0}^{n} \alpha_i X_i \frac{\partial}{\partial X_i}$$

$$= \pi_*\Big(\sum_{j \neq i} \alpha_j X_j \frac{\partial}{\partial X_j} + \alpha_i X_i \frac{\partial}{\partial X_i}\Big)$$

$$= \sum_{j \neq i} \alpha_j\, x_j \frac{\partial}{\partial x_j} + \alpha_i \Big(- \sum_{j \neq i} x_j \frac{\partial}{\partial x_j}\Big)$$

$$= \sum_{j \neq i} (\alpha_j - \alpha_i)\, x_j \frac{\partial}{\partial x_j}$$

$$\Rightarrow \quad A_{p_i} = \begin{pmatrix} \alpha_0 - \alpha_i & 0 & & 0 \\ 0 & \alpha_1 - \alpha_i & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & & & \\ 0 & \cdots\ 0 & & \alpha_n - \alpha_i \end{pmatrix}$$

$\lrcorner$

According to the Bott residue formula, then

$$c_1(\mathbb{P}^n)^n = \sum_{i=0}^{n} \frac{(\operatorname{trace}(A_{p_i}))^n}{\det(A_{p_i})}$$

$$= \sum_i \frac{\big(\sum_{j \neq i} (\alpha_j - \alpha_i)\big)^n}{\prod_{j \neq i} (\alpha_j - \alpha_i)}$$

$$= \sum_i \frac{(-(n+1)\alpha_i)^n}{\prod_{j \neq i} (\alpha_j - \alpha_i)},$$