It follows that the singleton $\{\phi_1\}$ is a closed set, relative to τ.

Pf. Since $\phi_2 + W \in \tau$, and $\phi_1 \notin \phi_2 + W$, we have found an open containg ϕ_2 but not ϕ_1. $\Rightarrow$ ϕ_1 is closed. ⌟

Addition is τ-continuous, since the convexity of every $W \in \beta$ implies that

(7) $$(\psi_1 + \tfrac{1}{2}W) + (\psi_2 + \tfrac{1}{2}W) = (\psi_1 + \psi_2) + W$$

for any $\psi_1 \in \mathcal{D}(\Omega)$, $\psi_2 \in \mathcal{D}(\Omega)$.

Pf.
$$\psi_1 + \tfrac{1}{2}\phi_1 + \psi_2 + \tfrac{1}{2}\phi_2 = \psi_1 + \psi_2 + (\tfrac{1}{2}\phi_1 + \tfrac{1}{2}\phi_2) \in \psi_1 + \psi_2 + W$$
$$\psi_1 + \psi_2 + \phi \in \psi_1 + \psi_2 + W, \quad \Rightarrow \quad \psi_1 + \psi_2 + \tfrac{1}{2}\phi + \tfrac{1}{2}\phi \in \psi_1 + \tfrac{1}{2}W + \psi_2 + \tfrac{1}{2}W.$$
$$\Rightarrow (\psi_1 + \tfrac{1}{2}W) + (\psi_2 + \tfrac{1}{2}W) = (\psi_1 + \psi_2) + W.$$

$$\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \xrightarrow{f} \mathcal{D}(\Omega)$$
$$(\psi_1, \psi_2) \longmapsto \psi_1 + \psi_2.$$

Given an open set W' containing $\psi_1 + \psi_2$, then $\exists$ W s.t $\psi_1 + \psi_2 + W \subset W'$ since $\{\phi + W\}$ forms a base for τ.

$\Rightarrow$ $f^{-1}(W') \supset (\psi_1 + W \cdot \tfrac{1}{2}, \psi_2 + \tfrac{1}{2}W)$ $\Rightarrow$ This implies that $f^{-1}(W')$ is open. $\Rightarrow$ f is continuous. ⌟

To deal with scalar multiplication, pick a scalar α_0 and $\phi_0 \in \mathcal{D}(\Omega)$. Then

(8) $$\alpha\phi - \alpha_0\phi_0 = \alpha(\phi - \phi_0) + (\alpha - \alpha_0)\phi_0.$$

If $W \in \beta$, $\exists$ $\delta > 0$ s.t $\delta\phi_0 \in \tfrac{1}{2}W$. Consider $\tfrac{1}{2}W \cap \{\alpha\phi_0\}$