Γ $|L| + mC = \{ \sigma \otimes \tau \mid \sigma \in H^0(M, \mathcal{O}(L)),\ \tau \in H^0(M, \mathcal{O}(mC))$ s.t $(\tau = 0) = mC \}$. Since $M - C$ is open & dense and if $\sigma_1 \cdots \sigma_n$ are a base for $H^0(M, \mathcal{O}(L))$, $\sigma_1|_{M-C}, \cdots \sigma_n|_{M-C}$ are linearly independent. $\Rightarrow$ $\sigma_1 \otimes \tau, \cdots \sigma_n \otimes \tau$ are linearly independent on $M-C$ since $\tau \neq 0$ on $M-C$. $\Rightarrow$

$$\begin{array}{ccc} M & \longrightarrow & \mathbb{P}^n \\ \Psi & & \Psi \\ p & \longmapsto & [\sigma_1 \otimes \tau(p), \cdots \sigma_n \otimes \tau(p)] \end{array}$$

is an embedding on $M-C$, since $p \longmapsto [\sigma_1(p), \cdots \sigma_n(p)]$ is embedding on M. $\Rightarrow$ $p \xmapsto{\iota_{L'}} [\sigma_1 \otimes \tau(p), \cdots \sigma_n \otimes \tau(p), \cdots]$ is embedding on $M-C$. If $p \in C$ & $q \in M-C$, then $\sigma_1 \otimes \tau(p) = \cdots \sigma_n \otimes \tau(p) = 0$, & not all $\sigma_1 \otimes \tau(q), \cdots \sigma_n \otimes \tau(q)$ are zero, since not all $\sigma_i(q)$'s are zero, ($\because$ ι_L is embedding). ﺩ

On the other hand, since $L'|_C$ is trivial, any section $\sigma \in H^0(M, \mathcal{O}(L'))$ vanishing at a point of C vanishes identically along C; so $\iota_{L'}$ maps C to a point.

Γ Any bundle over $C \cong \mathbb{P}^1$ is determined by its Chern class, i.e., $c_1(L'|_C) \in H^2(C, \mathbb{Z})$, and $L' \cdot C = \deg(L'|_C) = \int_C c_1(L'|_C) = L' \cdot C = (L + mC) \cdot C = L \cdot C + mC \cdot C = m - m = 0$ $\Rightarrow$ $L'|_C$ is trivial $\Rightarrow$ If $\sigma \in H^0(M, \mathcal{O}(L'))$ vanishes at $p \in C$, $\deg \sigma|_C = \#(\sigma|_C = 0) > 0$. $\Rightarrow$ $L'|_C$ is not trivial, since $\deg \sigma|_C = \deg(L'|_C)$.
$\Rightarrow$ σ must vanish identically along C.

Suppose $p, q \in C$, distinct. Assume that $\sigma_1(p) = \alpha\, \sigma_2(p)$, $\alpha > 0$. $\Rightarrow$ $\sigma_1 - \alpha\sigma_2$ vanishes at p. $\Rightarrow$ By