Given a surface M and a curve C on M, when can we realize M as the blow-up $\tilde{N}_{x_0}$ of some surface N, with $C = \pi^{-1}(\{x_0\})$? Clearly, necessary conditions are that C be rational and that $C \cdot C = -1$; in fact, the following result says that these are also sufficient.

$\ulcorner$ Since $C \cong \mathbb{P}^1$, and, by referring to the definition of the rational normal curve, C is rational, I guess. As we saw on P475, $C \cdot C = -1$. $\lrcorner$

Castelnuovo-Enriques Criterion. Let M be an algebraic surface, $C \subset M$ a smooth rational curve on M of self-intersection -1. Then there exists a smooth algebraic surface N and a map $\pi : M \longrightarrow N$ such that $M \xrightarrow{\pi} N$ is the blow-up of N at $p_0 \in N$, and $C = \pi^{-1}(p_0)$.

Proof. The proof here is along the lines of the Kodaira embedding theorem, but with a twist: we want to find a map $f : M \longrightarrow \mathbb{P}^m$ that is one to one away from C, maps C to a point, and has smooth image. Accordingly, we look first for a line bundle $L \longrightarrow M$ that is sufficiently positive away from C to have global sections, but whose restriction to C is trivial.

To find such a bundle, we start with a very ample line bundle L on M; choosing L sufficiently