$\Rightarrow$ By the result on P734, $T_p(F) \cap F$ is of rank $n-1$, equivalently, the cone through an 0-plane $\{p\} \subset T_p(F) \cap F \subset \mathbb{P}^{n-1}$ over a quadric of rank $n-1$ in $\mathbb{P}^{n-2}$. ⌟

In general, if F has rank k and singular set Λ_{n-k}, then every tangent hyperplane to F contains Λ, and $\mathcal{G}$ maps F to a smooth quadric in the subspace $\mathbb{P}^{k-1*} \subset \mathbb{P}^{n*}$ of hyperplanes containing Λ.

⌈ Since $\mathcal{G} : F \longrightarrow \mathbb{P}^{n*}$

$$p = [a_0, \cdots a_n] \longmapsto T_p(F) = ({}^tX Q a = 0),$$

if $\overset{p}{\underset{\shortparallel}{}} a \in \Lambda_{n-k}$, then $T_p(F) \ni a$. $\Rightarrow$ $T_p(F) \supset \Lambda = \Lambda_{n-k}$.

By the note P732, by changing the coordinates, we may assume
$$Q = \begin{array}{c} \\ k \end{array}\!\!\begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{k} & 0 \\ 0 & 0 \ \ddots \ 0 \end{pmatrix}.$$

$\Rightarrow$ $F = \sum_{i=0}^{k-1} X_i^2$

$\Rightarrow$ $\mathcal{G}(p) = \mathcal{G}([a_0, \cdots a_n]) = \{ \sum_{i=0}^{k-1} a_i X_i = 0 \} \in \mathbb{P}^{k-1*} \subset \mathbb{P}^{n*}$

$\Rightarrow$ Since $F|_{\mathbb{P}^{k-1}}$ is smooth, by the argument above, $\mathcal{G}(F|_{\mathbb{P}^{k-1}})$ is smooth quadric in $\mathbb{P}^{k-1*}$.

Linear Spaces on Quadrics

One fascinating aspect of quadrics is the behavior of the linear spaces lying on them. This