$$= \sum a^{\nu}_{jk} x_j \frac{\partial}{\partial x_{2k-1}} + \sum b^{\nu}_{jk} x_j \cdot \frac{\partial}{\partial x_{2k}} + [\partial],$$

so the index of v' at P_ν is $(-1)^{n(n-1)/2}$ times the sign of the determinant of A_{P_ν}. Thus by the Hopf index theorem (to be proved in the next section)

$$\chi(M) = \sum (-1)^{n(n-1)/2} \operatorname{sgn} \det(A_{P_\nu}),$$

and so we have

Gauss-Bonnet Formula III. $C_n(M) = \chi(M)$.

$$\Gamma v'(z) = \frac{1}{2} \sum (a^{\nu}_{jk} + \sqrt{-1}\, b^{\nu}_{jk})(x_{2j-1} + \sqrt{-1}\, x_{2j})\left(\frac{\partial}{\partial x_{2k-1}} - \sqrt{-1}\frac{\partial}{\partial x_{2k}}\right)\frac{1}{2}$$
$$+ \frac{1}{2} \sum (a^{\nu}_{jk} - \sqrt{-1}\, b^{\nu}_{jk})(x_{2j-1} - \sqrt{-1}\, x_{2j})\left(\frac{\partial}{\partial x_{2k-1}} + \sqrt{-1}\frac{\partial}{\partial x_{2k}}\right)\frac{1}{2}$$
$$= \frac{1}{2} \sum \frac{1}{2}(a^{\nu}_{jk} x_{2j-1} + \sqrt{-1}\, b^{\nu}_{jk} x_{2j-1} + \sqrt{-1}\, a^{\nu}_{jk} x_{2j} - b^{\nu}_{jk} x_{2j})\left(\frac{\partial}{\partial x_{2k-1}} - \sqrt{-1}\frac{\partial}{\partial x_{2k}}\right)$$
$$+ \frac{1}{4} \sum (a^{\nu}_{jk} x_{2j-1} - \sqrt{-1}\, b^{\nu}_{jk} x_{2j-1} - \sqrt{-1}\, a^{\nu}_{jk} x_{2j} - b^{\nu}_{jk} x_{2j})\left(\frac{\partial}{\partial x_{2k-1}} + \sqrt{-1}\frac{\partial}{\partial x_{2k}}\right) + [\partial]$$
$$= \frac{1}{4} \sum a^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k-1}} + \cancel{\sqrt{-1}\, b^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k-1}}} + \cancel{\sqrt{-1}\, a^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k-1}}} - b^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k-1}}$$
$$- \cancel{\sqrt{-1}\, a^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k}}} + b^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k}} + a^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k}} + \cancel{\sqrt{-1}\, b^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k}}}$$
$$+ a^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k-1}} - \cancel{\sqrt{-1}\, b^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k-1}}} - \cancel{\sqrt{-1}\, a^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k-1}}} - b^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k-1}}$$
$$+ \cancel{\sqrt{-1}\, a^{\nu}_{jk} x_{2j-1}} \frac{\partial}{\partial x_{2k}} + b^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k}} + a^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k}} - \cancel{\sqrt{-1}\, b^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k}}} + [\partial]$$
$$= \frac{1}{4} \sum 2 a^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k-1}} - 2 b^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k-1}}$$
$$+ \frac{1}{4} \sum 2 b^{\nu}_{jk} x_{2j-1} \frac{\partial}{\partial x_{2k}} + 2 a^{\nu}_{jk} x_{2j} \frac{\partial}{\partial x_{2k}} \qquad + [\partial]$$