is injective if $p, q$ impose linearly independent conditions for any $q$ infinitely near $p$.

Thus we proved that $\iota_{\tilde{C}}$ is homeomorphic onto its image and $\iota_{\tilde{C}*}(p)$ is injective for all $p \in \tilde{\mathbb{P}}^2$, which implies that $\iota_{\tilde{C}}$ is embedding. ⌟

Before proceeding to study the geometry of $S$, we make one observation. Recall that a smooth quadric surface $S \subset \mathbb{P}^3$ may be obtained by blowing up two points $q_1, q_2$ on $\mathbb{P}^2$, and blowing down the proper transform in $\tilde{\mathbb{P}}^2_{q_1,q_2}$ of the line $\overline{q_1 q_2}$.

⌜ See P480 ⌟

The reader may wish to verify, by the techniques of the preceding argument, that the linear system $|\pi^* 2H - E_1 - E_2|$ on $\tilde{\mathbb{P}}^2_{q_1 q_2}$ – corresponding to cubic curves in $\mathbb{P}^2$ passing through $q_1$ and $q_2$ – does indeed give a map of $\tilde{\mathbb{P}}^2_{q_1 q_2}$ onto a quadric surface in $\mathbb{P}^3$, one-to-one except along the proper transform of $\overline{q_1 q_2}$.

⌜ $\tilde{C} = \pi^* 2H - E_1 - E_2$

$\tilde{C} \cdot \tilde{C} = \pi^* 2H \cdot \pi^* 2H + E_1 \cdot E_1 + E_2 \cdot E_2 = 4 - 2 = 2$

In brief, the system $|\tilde{C}|$ consists exactly of the curves $\pi^* C - E_1 - E_2$, where $C$ is a conic curve containing $q_1, q_2$. $\Rightarrow$ $\dim H^0(\mathbb{P}^2, \mathcal{O}(2H)) = 6$. $\Rightarrow$ Since $q_1 \& q_2$ impose linearly independent conditions, $|\tilde{C}| = 6 - 2 - 1 = 3$.