$S$ is the zero locus of $F_0, F_1, \cdots F_n$.

Clearly $S \subset \Delta \times \mathbb{P}^1$.

If $V_\lambda$ is the singular locus of $D_\lambda$ in $\Delta$,

$V_\lambda = \bigcap_{i=0}^{n} \{F_i|\Delta \times \{\lambda\} = 0\}$.

Question : $V \cap \Delta = \pi(\bigcap_{i=0}^{n}\{F_i = 0\}) = \pi(S)$

$P \in V \cap \Delta. \Rightarrow P \in D_\lambda \Rightarrow P_\lambda = P$

$\Rightarrow F_i(P_\lambda, \lambda) = 0$ as we saw above for each $i$.

$\Rightarrow (P_\lambda, \lambda) \in \{F_i = 0\} \Rightarrow (P_\lambda, \lambda) \in \bigcap^{n} \{F_i = 0\}$

$\Rightarrow P_\lambda \in \pi(\bigcap\{F_i = 0\}) \Rightarrow V \cap \Delta \subset \pi(\bigcap_{i=0}^{n}\{F_i = 0\})$.

$Q \in \pi(\bigcap_{i=0}^{n}\{F_i = 0\}) \Rightarrow \exists (Q, r) \in \bigcap_{i=0}^{n}\{F_i = 0\}$

$\Rightarrow (Q, r) \in \{F_i = 0\}$ for all $i$, $\Rightarrow F_i(Q, r) = 0$ for all $i$

$\Rightarrow f(Q) + r\, g(Q) = 0, \quad \frac{\partial f}{\partial z_i}(Q) + r \frac{\partial g}{\partial z_i}(Q) = 0$ for $i = 1 \dots n$.

$\Rightarrow Q \in D_r$ and $Q$ singular point of $D_r$

$\Rightarrow Q \in V \cap \Delta$.

$\Rightarrow \pi(\bigcap\{F_i = 0\}) \subset V \cap \Delta$.

$\Rightarrow$ Thus $\pi(\bigcap\{F_i = 0\}) = V \cap \Delta$.

Then by P34 (Proper Mapping Theorem), $V$ is an analytic subvariety of $\Delta$.

More precisely, $S \subset \Delta \times \mathbb{P}^1$ $S$ is analytic variety

$\Rightarrow$ Question : $\pi(S)$ is an analytic variety ?

$$\begin{array}{c} S \subset \Delta \times \mathbb{P}^1 \\ \pi \downarrow \\ \Delta \end{array}$$

$\Rightarrow$ Obviously, $\pi$ is a proper mapping.

$K$ compact set in $\Delta$

$\Rightarrow \pi^{-1}(K)$ is compact, cl. $K \times \mathbb{P}^1$.

$\Rightarrow K \times \mathbb{P}^1 \cap (\bigcap\{F_i = 0\})$ closed $\Rightarrow$ compact $\Rightarrow \pi|S$ is proper.