7. Kähler Manifolds.

The Kähler Condition

Let M compact complex with Hermitian metric ds^2, and suppose that in some open set $U \subset M$, ds^2 is Euclidean; that is $\exists$ local holomorphic coordinates $z = (z_1, z_2 \cdots z_n)$ s.t

$$ds^2 = \sum dz_i \otimes d\bar{z}_i .$$

Write $z_i = x_i + i y_i$; one may directly verify that for a differential form

$$\varphi = \sum \varphi_{I\bar{J}}\, dz_I \wedge d\bar{z}_J \quad \text{compactly supported in } U,$$

$$\Delta_{\bar\partial} \varphi = -2 \sum_{I J i} \frac{\partial^2}{\partial z_i \partial \bar{z}_i} \varphi_{I\bar{J}}\, dz_I \wedge d\bar{z}_J .$$

$$= -\frac{1}{2} \sum_{I,J,i} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right) \varphi_{I\bar{J}}\, dz_I \wedge d\bar{z}_J .$$

$$= \frac{1}{2} \Delta_d(\varphi) .$$

i.e the $\bar\partial$ - Laplacian is equal to the ordinary d-Laplacian in U up to a constant (cf. Section 6. p83). Of course, very few compact complex manifolds have everywhere Euclidean metrics, but as it turns out in order to insure the identity

$$\Delta_{\bar\partial} = \frac{1}{2} \Delta_d . \quad \text{on a complex manifold,}$$

it is sufficient that the metric approximate the Euclidean metric to order 2 at each point. This is the Kähler condition, and we will spend the greater part of this section discussing the condition and its consequences.

We start by giving three alternate forms of the Kähler condition. Again, let $ds^2 = \sum h_{i\bar{j}}\, dz_i \otimes d\bar{z}_j = \sum \varphi_i \otimes \bar{\varphi}_i$.