「 $\dim H^0(\mathbb{P}^2, \mathcal{O}(3H)) = \binom{2+3}{2} = {}_5C_2 = \frac{5\cdot 4}{2} = 10$

$\Rightarrow \quad |3H| = 10 - 1 = 9. \qquad H^0(\mathbb{P}^2, \mathcal{O}(3H)) = \langle \sigma_1, \sigma_2, \cdots \sigma_{10} \rangle.$

$\Rightarrow \quad a_1\sigma_1 + \cdots + a_{10}\sigma_{10} = \sigma \quad \Rightarrow \quad \sigma(p) = 0 = a_1\sigma_1(p) + \cdots + a_{10}\sigma_{10}(p)$

$\Rightarrow \{\sigma(p_i) = 0\}_{i=1,\cdots 6}$ is linearly independent $\Leftrightarrow$ the space of $\{(a_1, \cdots a_{10})\}$'s satisfying the above conditions has dimension 4. $\Leftrightarrow \dim|\tilde{C}| = 4 - 1 = 3.$ 」

This, and the last two assertions as well, will follow from the

Lemma. Eight points $p_1, p_2, \cdots p_8 \in \mathbb{P}^2$ fail to impose independent conditions on cubics only if either

1. All eight lie on a conic curve; or
2. Five of the points p_i are collinear.

Proof. The first step in the proof is to show that seven points $p_1, p_2, \cdots p_7 \in \mathbb{P}^2$ fail to impose independent conditions on cubics only if five are collinear.
To see this, we argue as follows: Assume that $p_1, p_2, \cdots p_7$ fail to impose independent conditions on cubics. Then for some point p_i, any cubic containing the other six will contain p_i; reordering, we may take p_i to be p_1.

「 If not so, then for each p_i, $\exists$ a conic curve σ_i s.t $\sigma_i(p_j) = 0$, $j \neq i$, and $\sigma_i(p_i) \neq 0$.