$H^1(S, Q) = 0$, and $\qquad H_1(S) = H_1(\tilde{S}) \oplus H_1(\mathbb{P}^2) \oplus \cdots$.

(By $H_i(\tilde{M}) = H_i(M) \oplus H_i(E)$, $i > 0$ on P474,

$H_i(S) = H_i(\tilde{S}) \oplus H_i(D_1) \oplus \cdots \oplus H_i(D_6)$.)

$$\Rightarrow \quad H_2(S) = H_2(\tilde{S}) \oplus H_2(\mathbb{P}^1) \oplus \cdots \oplus H_2(\mathbb{P}^1)$$

$$= H_2(\tilde{S}) \oplus \mathbb{C}^6.$$

$\Rightarrow \quad b_2(S) = b_2(\tilde{S}) + 6 \quad \Rightarrow \quad b_2(\tilde{S}) = b_2(S) - 6 = b_2(S_0) - 6 = 7 - 6 = 1$ (The proof of $b_2(S) = 7$ is below.)

By Noether's formula,

$$1 = \chi(\mathcal{O}_{S_0}) = \frac{1}{12}(K_{S_0} \cdot K_{S_0} + \chi(S_0)).$$

Since
$$K_{S_0} \cdot K_{S_0} = +[H \cap S_0] \cdot [H \cap S_0]$$
$$= \#((H \cap S_0) \cap (H \cap S_0))$$
$$= \#(S_0 \cap (H \cap H)) = \#(S_0 \cap \ell) = 3,$$

$12 = 3 + \chi(S_0)$, and $\chi(S_0) = 9$.

$\Rightarrow \quad \chi(S) = \chi(S_0) = 9 = h^0(S) + h^2(S) + h^4(S)$

since $h^1(S) = h^3(S) = 0$.

$\Rightarrow$ Since $h^0(S) = h^4(S) = 1$, $\quad h^2(S) = 7$.

$\Rightarrow \quad b_2(S) = b^2(S) = 7$. $\square$

Note also that if $\pi : S \longrightarrow \tilde{S}$ is the blowing-down map, then

$$K_S = \pi^* K_{\tilde{S}} + D_1 + D_2 + \cdots + D_6,$$

and since K_S is negative, we deduce that for any curve $D \subset \tilde{S}$,

$$D \cdot K_{\tilde{S}} = \pi^* D \cdot (K_S - D_1 - \cdots - D_6)$$
$$\leq \pi^* D \cdot K_S < 0;$$

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