If $\deg(L-p) = \deg L - 1 > \deg K_S$, then $H^1(S, \mathcal{O}(L-p)) = 0$ and it follows that

$$H^0(S, \mathcal{O}(L)) \longrightarrow L_p \longrightarrow 0,$$

that is, the complete linear system of a line bundle of degree greater than $\deg K_S + 1$ has no base points.

Γ $\deg(L-p) = \deg L - \deg p = \deg L - 1 > \deg K_S$.

$\Rightarrow$ $H^1(S, \mathcal{O}(L-p)) = H^1(S, \Omega^1((L-p)\otimes K_S^*)) = 0$ by Kodaira vanishing theorem.

$\Rightarrow$ We get the exact sequence.

$$H^0(S, \mathcal{O}(L-p)) \longrightarrow H^0(S, \mathcal{O}(L)) \longrightarrow \underset{\substack{\| \\ L_p}}{H^0(S, L_p)} \rightarrow \underset{\substack{\| \\ H^1(S, \mathcal{O}(L-p))}}{0}$$

$)) = 0$.

$\mathbb{P}(H^0(S, \mathcal{O}(L))) \longleftrightarrow |D|$, $L = [D]$.

Suppose all $\forall \sigma \in H^0(S, \mathcal{O}(L))$ have zero at $p \in S$.

$\Rightarrow$

$H^0(S, \mathcal{O}(L)) \longrightarrow L_p \longrightarrow 0$ is not true since $\sigma \longmapsto \sigma(p) = 0$ is not surjective.

Assume that the complete linear system has a base point $p \in S$. $\Rightarrow$ Since $\mathbb{P}(H^0(S, \mathcal{O}(L))) \overset{1-1}{\Leftrightarrow} |D|$, all $\sigma \in H^0(S, \mathcal{O}(L))$ become zero at p. $\Rightarrow$ By the above argument, contradiction. ♪

Moreover, if $\deg L > \deg K_S + 2$, then from the exact sequences

$$0 \longrightarrow \mathcal{O}(L-p-q) \longrightarrow \mathcal{O}(L) \xrightarrow{r_{p,q}} L_p \oplus L_q \rightarrow 0,$$
$$0 \longrightarrow \mathcal{O}(L-2p) \longrightarrow \mathcal{O}(L) \xrightarrow{d_p} T_p'^* \otimes L_p \rightarrow 0,$$