Consider an open covering $\{V_x\}_{x\in M}$.
Then since M is paracompact, $\exists$ a locally finite refinement of $\{V_x\}_{x\in M}$. We call it $\underline{U}'$.

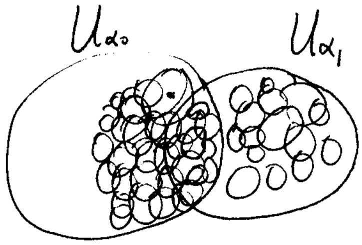
Actually, here we show that in the sequence

$$0 \longrightarrow C^p(\underline{U}, \mathcal{E}) \xrightarrow{\alpha} C^p(\underline{U}, \mathcal{F}) \xrightarrow{\beta} C^p(\underline{U}, \mathcal{G}) \longrightarrow 0,$$

given $\sigma \in C^p(\underline{U}, \mathcal{G})$, $\exists$ an open covering $\underline{U}'$ s.t

$$r_{U_{\alpha_0}\cap\cdots\cap U_{\alpha_p},\, U'_{\beta_0}\cap\cdots\cap U'_{\beta_p}}(\sigma_{\alpha_0\cdots\alpha_p}) = \beta(\tau_{\beta_0\cdots\beta_p}). \text{ where}$$

$$\tau_{\beta_0\cdots\beta_p} \in U'_{\beta_0}\cap\cdots\cap U'_{\beta_p}.$$



To see that

$$H^p(M, \mathcal{F}) \xrightarrow{\beta^*} H^p(M, \mathcal{G}) \xrightarrow{\delta^*} H^{p+1}(M, \mathcal{E}) \text{ is exact,}$$

we will use the above construction.

Let $\sigma \in C^p(\underline{U}, \mathcal{G})$ with $\delta\sigma = 0$ and $\delta^*\sigma = 0$ in $H^{p+1}(\underline{U}, \mathcal{E})$. $\;{}''\delta^*\rho\sigma$

Then $\exists$ $\tau \in C^p(\underline{U}', \mathcal{F})$ s.t $\beta\tau = \rho\sigma$. and $\mu \in C^{p+1}(\underline{U}'', \mathcal{E})$ s.t $\alpha\mu = \rho\delta\tau$ --- ①

$\Rightarrow$ By definition, $\mu = \delta\overset{*}{\rho}\sigma$. In $H^{p+1}(\underline{U}'', \mathcal{E}) = \dfrac{\operatorname{Ker}\delta}{\operatorname{Im}\delta}$
$\qquad\qquad\qquad\qquad\quad\overset{\shortparallel}{0}$

So $\mu = \delta\nu$ for some $\nu \in C^p(\underline{U}'', \mathcal{E})$.
Then $\rho\tau - \alpha\nu$ is a cocycle in $C^p(\underline{U}'', \mathcal{F})$ since

$$\delta(\rho\tau - \alpha\nu) = \rho\delta\tau - \alpha\delta\nu \overset{①}{=} \alpha\mu - \alpha\mu = 0.$$

$\Rightarrow$ $\beta(\rho\tau - \alpha\nu) = \rho\beta\tau = \rho\rho\sigma$ shows $\rho^2\sigma \in \beta^*(H^p(\underline{U}'', \mathcal{F}))$