$\ulcorner$ $C_r(S)(Z_r) = C_r(S|_{Z_r})(Z_r)$. since $C_r(S) \in H^{2r}_{DR}(Z_k)$
and $i^* C_r(S) \in H^{2r}_{DR}(Z_r)$.

$$\begin{tikzcd} i^*Q \arrow[r] \arrow[d] & Q \arrow[d] \\ Z_r \arrow[r, "i"] & Z_k = G(k, k+1) \end{tikzcd}$$

Note $C_r(S) \in H^{2r}_{DR}(G(k,n)) \Rightarrow C_r(S|_{Z_k}) \in H^{2r}_{DR}(Z_k)$.
rigorously speaking.

$$C_r(S|_{Z_r})\cdot(Z_r) = (-1)^r (f^*\omega)^r(Z_r) = (-1)^r \omega^r(f_* Z_r)$$
$$= (-1)^r \omega^r(G(1, r+1)) = (-1)^r \omega^r(\mathbb{P}^r) = (-1)^r. \quad \lrcorner$$

Note that by the relation giving the Chern classes of a dual bundle

$$C_r(S^*) = (-1)^r C_r(S) = \sigma^*_{1,\cdots,1}$$

and via the isomorphism $(S^* \to G(n-k, n)) \cong (Q \to G(k,n))$, we see that

$$C_r(Q) = \sigma^*_r ,$$

where Q is the universal quotient bundle.

$\ulcorner$ $C(S^* \otimes S) = 1 \Rightarrow$ not useful right now
By P408, fact 4, $C_r(S^*) = (-1)^r C_r(S) = \sigma^*_{1,\cdots,1}$.
$= C_r(f^*Q) = f^*(C_r(Q)) = \sigma^*_{1,\cdots,1}$.
$\Rightarrow$ We have only to show that $f_* \sigma_{1,\cdots,1} = \sigma_r$. i.e
$* \sigma_r = \sigma_{1,1,\cdots,1}$, or $* \sigma_{1,1,\cdots,1} = \sigma_r$ since $* = f$ is an isomorphism.

$a_1 = 1, \cdots a_r = 1, \quad a_{r+1} = \cdots = a_k = 0$
$a^*_{a_1} = a^*_1 \geq 1 \quad \cdots \quad a^*_{a_r} = a^*_1 \geq r. \quad \Rightarrow a^*_1 = r. \quad a^*_0 = 0. \quad a^*_2 = 0 \cdots = a^*_k$