Proof. It is clear that the image of $E_1 \xrightarrow{\partial} E_0 = \mathcal{O}$ is just the ideal I, so that $H_0(E.(f)) \cong \mathcal{O}/I$. We shall prove by induction on r that the higher homology is zero.

$\ulcorner$ $E_1 = \mathcal{O} \otimes_{\mathbb{C}} \Lambda^1 \mathbb{C}^r = \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C}^r$

$$\begin{array}{ccccc} E_1 & \xrightarrow{\partial} & E_0 = \mathcal{O} \otimes_{\mathbb{C}} \mathbb{C} & \cong & \mathcal{O} \\ \Downarrow & & \Downarrow & & \Downarrow \\ 1 \otimes e_i & \longmapsto & f_i \quad 1 \otimes_{\mathbb{C}} 1 & \leftrightarrow & f_i \end{array} \Rightarrow \partial(E_1) = I \subset \mathcal{O}$$ $\lrcorner$

In case $r=1$ the Koszul complex is $0 \to \mathcal{O} \xrightarrow{f_1} \mathcal{O}$ since $f_1 \neq 0$, and the result is clear.

$\ulcorner$ $r = 1$, $E_K = \mathcal{O} \otimes_{\mathbb{C}} \Lambda^K \mathbb{C} = \begin{cases} 0 & K \geq 2 \\ \mathcal{O} & K = 1 \\ \mathcal{O} & K = 0 \end{cases}$

$$\begin{array}{ccccccc} & & \mathcal{O} & & \mathcal{O} & & \\ & & \| & & \| & & \\ 0 & \longrightarrow & E_1 & \xrightarrow{\partial} & E_0 & \longrightarrow & \\ & & \Downarrow & & \Downarrow & & \\ & & 1 \otimes e & \longmapsto & f (= f_1) & & \end{array} \Rightarrow \partial(g \cdot 1 \otimes e) = g \cdot f$$

$f \neq 0 \Leftrightarrow f$ is not zero divisor in $\mathcal{O}$.

$\Rightarrow H_1(E.(f)) = \ker \partial = 0$ since f is not zero divisor. $\lrcorner$

Now we assume the result for $r-1$, and let $F_K \subset E_K$ be induced by the inclusion $\Lambda^K \mathbb{C}^{r-1} \subset \Lambda^K \mathbb{C}^r$, where $\mathbb{C}^{r-1}$ is spanned by $e_1, \cdots e_{r-1}$.

$\ulcorner$ $F_K = \mathcal{O} \otimes_{\mathbb{C}} \Lambda^K \mathbb{C}^{r-1} \subset \mathcal{O} \otimes_{\mathbb{C}} \Lambda^K \mathbb{C}^r$. $\lrcorner$