$$\theta_e^1 = \partial h_1 \cdot h_1^{-1} = \partial(A\,{}^t\bar{A}) \cdot {}^t\bar{A}^{-1} A^{-1}$$
$$\theta_e^2 = \partial h_2 \cdot h_2^{-1} = \partial(B\,{}^t\bar{B}) \cdot {}^t\bar{B}^{-1} B^{-1}$$

$$\Rightarrow \quad \bar{\partial} B \cdot B^{-1} + B\,\theta_{v'}^2\, B^{-1} = -B\,{}^t\bar{B}\,(\partial\,{}^t\bar{B}^{-1})\,B^{-1} \quad \text{——} \ ①$$
$$\bar{\partial} B \cdot B^{-1} + B\,\bar{\partial} g\, g^{-1} B^{-1} + B g\,\theta_v^1\, g^{-1} B^{-1} = -B g\,{}^t\bar{g}\,{}^t\bar{B}\,\partial\,{}^t\bar{B}^{-1}\,{}^t\bar{g}^{-1} g^{-1} B^{-1}$$
$$- B g\,{}^t\bar{g}\,\partial\,{}^t\bar{g}^{-1} g^{-1} B^{-1} \quad \text{—} \ ②$$

$\Rightarrow$ ① $-$ ② $\Rightarrow$

$$\theta_{v'}^2 = \bar{\partial} g \cdot g^{-1} + g\,\theta_v^1\, g^{-1} - {}^t\bar{B}\,\partial\,{}^t\bar{B}^{-1} + g\,{}^t\bar{g}\;{}^t\bar{B}\,\partial\,{}^t\bar{B}^{-1}\,{}^t\bar{g}^{-1} g^{-1}$$
$$+ g\,{}^t\bar{g}\,\partial\,{}^t\bar{g}^{-1} g^{-1}$$

If $g^{-1} = {}^t\bar{g}$, then we get

$$\theta_{v'}^2 = dg\, g^{-1} + g\,\theta_v^1\, g^{-1}.$$

The condition $g^{-1} = {}^t\bar{g}$ implies that $\langle\ \rangle_1 = \langle\ \rangle_2$, for

$$\langle v_i', v_j' \rangle_1 = \langle g_{il} v_l,\ g_{jk} v_k \rangle = g_{il}\,\bar{g}_{jk} = (g\,{}^t\bar{g})_{ij} = \delta_{ij}$$

Don't misunderstand that $\tilde{\tau}$ is a tensor invariant of <u>the metric</u>, as $\tilde{\tau}$ is a tensor independent of a metric.

Note that by our calculation, if $v_1', \cdots v_n'$ is any other frame for T', not necessarily unitary but with φ, θ, and θ^* as before, we still have

$$\tilde{\tau} = \sum_i \left(d\varphi_i' - \sum \theta_{ij}'^* \wedge \varphi_j'\right) \otimes v_i'.$$

⌈ Let $v_i' = \sum g_{ij} v_j$. $\Rightarrow$ $\varphi' = g^* \varphi$. where $g^* = {}^t g^{-1}$

$$\Rightarrow \theta^* = g^{*-1}\,\theta'^*\, g^* + dg^{*-1} \cdot g^*$$

$$\begin{aligned}\tau &= d\varphi - \theta^* \wedge \varphi \\ &= d(g^{*-1}\varphi') - (g^{*-1}\theta'^* g^* + dg^{*-1} \cdot g^*) \wedge g^{*-1}\varphi' \\ &= \cancel{(dg^{*-1})\,\varphi'} + g^{*-1} d\varphi' - g^{*-1}\,\theta'^* \wedge \varphi' - \cancel{dg^{*-1} \wedge \varphi'} \\ &= g^{*-1}\, d\varphi' - g^{*-1}\,\theta'^* \wedge \varphi'.\end{aligned}$$

$$\Rightarrow \tilde{\tau} = \sum \tau_i \otimes v_i = \sum g^{*-1}_{il}\, d\varphi_l' - g^{*-1}_{ik}\,\theta_{kl}'^* \wedge \varphi_l' \otimes$$