Line Bundles on Complex Tori

We will now give explicit descriptions of positive line bundles on a complex torus $M = V/\Lambda$. The fundamental observation, proved on P. 46, is simply that, since $H^1(\mathbb{C}^n, \mathcal{O}) = H^2(\mathbb{C}^n, \mathbb{Z}) = 0$, any line bundle on $V \cong \mathbb{C}^n$ is trivial.

See P46 and P27 'note'.

Thus if $L \longrightarrow M$ is any line bundle, the pull back $\pi^* L$ of L to V is trivial, and we can find a global trivialization

$$\varphi : \pi^* L \longrightarrow V \times \mathbb{C}.$$

Now for $z \in V$, $\lambda \in \Lambda$, the fibers of $\pi^* L$ at z and $z+\lambda$ are by definition both identified with the fibers of L at $\pi(z)$, and comparing the trivialization φ at z and $z+\lambda$ yields a linear automorphism of $\mathbb{C}$:

$$\mathbb{C} \xleftarrow{\varphi_z} (\pi^* L)_z = L_{\pi(z)} = (\pi^* L)_{z+\lambda} \xrightarrow{\varphi_{z+\lambda}} \mathbb{C}.$$

$$\begin{tikzcd}
\pi^* L \cong V \times \mathbb{C} \arrow[r] \arrow[d] & L \arrow[d] \\
V \arrow[r, "\pi"] & M = V/\Lambda
\end{tikzcd}$$

$$\varphi : \pi^* L \xrightarrow{\cong} V \times \mathbb{C}$$

$$(\pi^* L)_z = \pi^* L_{\pi(z)} = (\pi^* L)_{z+\lambda}$$