At this point, we have proved the Hodge theorem. The essential idea is to produce the Green's operator by a Hilbert-space trick, and then to use the basic estimate to show that it is a compact smoothing operator.

Actually, G is an integral operator of the form

$$(G\varphi)(x) = \int_M G(x,y)\, \varphi(y).$$

where $G(x,y)$ is a beautiful kernel on $M\times M$ with certain singularities along the diagonal. The Hilbert-space method has the disadvantage of not giving us the Green's operator in this form. If we were working with distributions rather than just L^2-forms, then we could produce $G(x,y)$ by solving a distributional equation of the type

$$\Delta_x G(x,y) = \delta_y + S_y,$$

where δ_x is a delta function at y and S_y is an operator of order $-\infty$. Such equations will be discussed in Section 1 of Chapter 3.

$\lceil\!\lceil$ $\bar\partial\Delta = \Delta\bar\partial \quad \Rightarrow \quad \bar\partial\Delta\varphi_m = \Delta\bar\partial\varphi_m$

$\Rightarrow \quad \bar\partial\lambda_m\varphi_m = \Delta(\bar\partial\varphi_m) = \lambda_m(\bar\partial\varphi_m)$

$\Rightarrow \qquad G\bar\partial\varphi_m = \frac{1}{\lambda_m}\bar\partial\varphi_m = \bar\partial G(\varphi_m)$

$$\bar\partial(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = \bar\partial\bar\partial^*\bar\partial$$

$$(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\bar\partial = \bar\partial\bar\partial^*\bar\partial$$

$$\bar\partial^*(\bar\partial\bar\partial^* + \bar\partial^*\bar\partial) = \bar\partial^*\bar\partial\bar\partial^* = (\bar\partial\bar\partial^* + \bar\partial^*\bar\partial)\bar\partial^*$$

$\Rightarrow \quad \bar\partial^*\Delta = \Delta\bar\partial^* \qquad \Rightarrow \quad G\bar\partial^* = \bar\partial^* G.$ $\rfloor\!\rfloor$