then $(\partial f/\partial z_1)(q) \neq 0$ and — taking z_2 as local coordinate on S near q, $z_1 = z_1(z_2)$ as a function of z_2 — we can write

$$f(z_1(z_2), z_2) \equiv 0$$

so by the chain rule

$$\frac{\partial f}{\partial z_2} + \frac{\partial f}{\partial z_1} \cdot \frac{\partial z_1}{\partial z_2} \equiv 0 \text{ on } S.$$

Since $\left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}\right) \neq 0$ ($\because$ S is smooth and f is a defining function of S locally), we may have a point $q \in S$ with $\partial f/\partial z_2(q) \neq 0$. If not, $\partial f/\partial z_2 = 0$ on $\mathbb{C}^2$. $\Rightarrow$ f is a function of only z_1. Then we take $\partial f/\partial z_1$. Anyway, we choose a point $q \in S$ with $\partial f/\partial z_2(q) = 0$. See what happens next.

$\{(z_1, z_2) \in \mathbb{C}^2 \mid f(z_1, z_2) = 0\}$ is biholomorphic to an open subset of S. If $\partial f/\partial z_2(q) \neq 0$, by the implicit function theorem we have a holomorphic function $z_2(z_1)$ of z_1 s.t.

$$f(z_1, z_2(z_1)) = 0.$$

Consider the following

$$\mathbb{C}^2 \xrightarrow{T} \mathbb{C}^2$$

$$(z_1, z_2) \longmapsto (z_1, f(z_1, z_2)).$$

$\Rightarrow$ $\mathrm{Jac}(T) = \begin{pmatrix} 1, & 0 \\ \frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} \end{pmatrix}$ is not zero at q.

$\Rightarrow$ $\exists$ an inverse G locally s.t $G \circ T = \mathrm{id}$.

$\Rightarrow$ $G(z_1, f(z_1, z_2)) = (z_1, z_2)$ $\Rightarrow$ On $\{f(z_1, z_2) = 0\}$,