$$\epsilon\,(-1)^{(j,\,I\cup\{j\})}\, d\arg f_{i_1}\wedge\cdots\wedge d\arg f_{i_p}\wedge\Big(\bigwedge_{l\notin I\cup\{j\}}\tfrac{\sqrt{-1}}{2}\,df_l\wedge d\bar f_l\Big)\wedge\tfrac{\sqrt{-1}}{2}\,df_j\wedge d\bar f_j$$

$$=\epsilon\,(-1)^{(j,\,I\cup\{j\})}\, d\arg f_{i_1}\wedge\cdots\wedge d\arg f_{i_p}\wedge\Big(\bigwedge_{l\notin I}\tfrac{\sqrt{-1}}{2}\,df_l\wedge d\bar f_l\Big)$$

= the orientation of P_I

$$= d\arg f_{i_1}\wedge\cdots\wedge d\arg f_{i_p}\wedge\Big(\bigwedge_{l\notin I}\tfrac{\sqrt{-1}}{2}\,df_l\wedge d\bar f_l\Big)$$

$$\Rightarrow\quad \epsilon=(-1)^{(j,\,I\cup\{j\})}.$$

$$\Rightarrow\quad \partial P_I=\sum_{j\notin I}(-1)^{(j,\,I\cup\{j\})}P_{I\cup\{j\}}.$$

Here we assumed. $\partial P_I=\sum_{j\notin I}\epsilon_j\, P_{I\cup\{j\}}$, $dr\wedge\partial P_I \overset{\text{orientation}}{=} P_I$ and give an order on $I\cup\{j\}$ by its number order.

$$\sum_{\#I=p}\int_{P_I}\omega_{p-1,I}=\sum_{\#I=p}\int_{P_I}d\xi_{p,I}=\sum_{\#I=p}\int_{\partial P_I}\xi_{p,I}$$

$$=\sum_{\#I=p}\Big(\sum_{j\notin I}\int_{P_{I\cup\{j\}}}(-1)^{(j,\,I\cup\{j\})}\xi_{p,I}\Big)$$

$$=\sum_{\#J=p+1}\sum_{j\in J}\int_{P_J}(-1)^{(j,\,J)}\xi_{p,\,J-\{j\}}.\quad\cdots\ (*)$$

$(\delta\xi)_J=\sum(-1)^{(j,J)-1}\xi_{p,\,J-\{j\}}$, where $(j\ J)$ is the position from the front of the index j.

$$(*)=\sum_{\#J=p+1}\sum_{j\in J}\int_{P_J}(-1)^{(j,J)}\xi_{p,\,J-\{j\}}=\sum_{\#J=p+1}\sum_{j\in J}\int_{P_J}(-1)\qquad\xi_{p,\,J-\{j\}}$$

$$=\sum_{\#J=p+1}\sum_{j\in J}\int_{P_J}(-1)\qquad\xi_{p,\,J-\{j\}}=\sum_{\#J=p+1}\sum_{j\in J}\int_{P_J}(-1)\qquad\xi_{p,\,J-\{j\}}=(-1)^{p}\int_{P}\sum_{\#J=p+1}\sum_{j\in J}(-1)\quad\xi_{p,\,J-\{j\}}$$