From the identity $\Lambda\bar\partial - \bar\partial\Lambda = -\sqrt{-1}\,\partial^*$, $\quad\Lambda\bar\partial = \bar\partial\Lambda$,
since $\partial^*\varphi = 0$, $\quad \varphi \in C^\infty(\mathbb{C}^n)$.

$$\Rightarrow \sqrt{-1}\,\partial\Lambda\bar\partial - \bar\partial^*\bar\partial = \sqrt{-1}\,\partial\bar\partial\Lambda - \bar\partial^*\bar\partial = \sqrt{-1}\,\Lambda\partial\bar\partial.$$

$$\Rightarrow \sqrt{-1}\,\partial\Lambda\bar\partial - \sqrt{-1}\,\Lambda\partial\bar\partial = \bar\partial^*\bar\partial + \bar\partial\,\underset{(=0)}{\bar\partial^*} = \frac{1}{2}\Delta$$

$$\Rightarrow \text{If } \bar\partial T = 0, \text{ then } \Delta T = 0.$$

Thus, $\bar\partial T = 0 \Rightarrow \Delta T = 0$, and so $T = T_f$ for some $f \in C^\infty(U)$ by the preceding lemma.

We did already prove this.

But then $0 = \bar\partial T_f = T_{\bar\partial f} \Rightarrow \bar\partial f = 0$, and $f \in \mathcal{O}(U)$.

Q.E.D.

$$\bar\partial T_f(\varphi) = T_{\bar\partial f}(\varphi) \;?\;, \quad \varphi \in A_c^{2n-1}(\mathbb{C}^n) \supset A_c^{n,n-1}(\mathbb{C}^n).$$

$$\begin{aligned} &\quad\;\;\| \\ &- T_f(\bar\partial\varphi) = -\int_{\mathbb{C}^n} f \wedge \bar\partial\varphi = -\int_{\mathbb{C}^n} \bar\partial(f\varphi) + \int_{\mathbb{C}^n} \bar\partial f \wedge \varphi \\ &= -\int_{\mathbb{C}^n} d(f\varphi) + \int_{\mathbb{C}^n} \bar\partial f \wedge \varphi \\ &= \int_{\mathbb{C}^n} \bar\partial f \wedge \varphi = T_{\bar\partial f}(\varphi). \end{aligned}$$

Finally, we will tie up the remaining loose end in Section 6 of Chapter 0 on the proof of the Hodge theorem. Namely, referring to Regularity Lemma I in the