Γ $\mathcal{D}(\Omega) = \bigcup_{i=1}^{\infty} \mathcal{D}_{K_i}$ where $\mathcal{D}_{K_i}$ is closed and has nonempty interior.

$\Rightarrow$ $\mathcal{D}_{K_i}^c$ is open & dense. $\Rightarrow$ $\mathcal{D}(\Omega)^c = \phi = \bigcap \mathcal{D}_{K_i}^c$

If $\mathcal{D}(\Omega)$ is metrizable, by Baire's theorem, $\bigcap \mathcal{D}_{K_i}^c$ is non-empty. $\Rightarrow$ $\mathcal{D}(\Omega)$ is not metrizable. ⌟

Return to P367.

Examples.

1. If $\psi(x)$ is a locally L^1 function (see p185. F.A.) on $\mathbb{R}^n$, then we may define a distribution T_ψ of order zero by

$$T_\psi(\varphi) = \int_{\mathbb{R}^n} \varphi(x)\,\psi(x)\,dx.$$

Here $dx = dx_1 \wedge \cdots \wedge dx_n$, and we always assume that $\mathbb{R}^n$ is oriented by this form.

Γ $|T_\psi(\varphi)| = \left|\int_{\mathbb{R}^n} \varphi(x)\,\psi(x)\,dx\right| \leq \left(\int_U |\varphi(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_U |\psi(x)|^2 dx\right)^{\frac{1}{2}}$

$= \|\varphi\|_{L^2} \left(\int_U |\psi(x)|^2 dx\right)^{\frac{1}{2}}$ where $\varphi \in \mathcal{D}_K$, and $U \supset K$.

$\Rightarrow$ $|T_\psi(\varphi)| < C_K \|\varphi\|_{L^2}$

$|T_\psi(\varphi)| = \left|\int_{\mathbb{R}^n} \varphi(x)\,\psi(x)\,dx\right| \leq \int_{\mathbb{R}^n} |\varphi(x)\,\psi(x)|\,dx \leq \int_U \|\varphi\|_0 \,|\psi(x)|\,dx$

$= \|\varphi\|_0 \int_U |\psi(x)|\,dx \leq C_K \|\varphi\|_0$, where $\varphi \in \mathcal{D}_K$ and $U \supset K$.

$\Rightarrow$ Since, to every compact $K \subset \mathbb{R}^n$, $\exists$ $N_K = 0$ and C_K s.t

$|T_\psi(\varphi)| \leq C_K \|\varphi\|_0$ for all $\varphi \in \mathcal{D}_K$,

$T_\psi \in \mathcal{D}'(\mathbb{R}^n)$, by Theorem 6.8 Rudin's F.A. P141. $\Rightarrow$ T_ψ