$n$-cells, ~~8~~ and so the cells appearing in the coboundary $\delta\Delta_\alpha^{n-k}$ of $\Delta_\alpha^{n-k}$ will be just the $k$-fold intersections $\Delta_j^{n-k+1} = \bigcap_{i \neq j} \Delta_i^n$ of the ~~8~~ cells $\Delta_i^n$,

i.e. the dual cells of the faces $\sigma_j^{k-1}$ of $\sigma^k$.

We claim that the basic relation $\sigma_j^{k-1} = \langle \sigma_0^0 \cdots \hat{\sigma_j^0} \cdots \sigma_k^0 \rangle$

$$\delta(\Delta_\alpha^{n-k}) = (-1)^{n-k+1} * (\partial\sigma_\alpha^k) = (-1)^{n-k+1} * \left(\sum_{j=0}^{k} (-1)^j \sigma_j^{k-1}\right) = (-1)^{n-k+1} \sum_{j=0}^{k} (-1)^j * \sigma_j^{k-1} = (-1)^{n-k+1} \sum_{j=0}^{k} (-1)^j \Delta_j^{n-k+1}$$

holds on the level of oriented cells. i.e that if $\sigma_j^{k-1}$ and $\Delta_j^{n-k+1}$ are oriented as the boundary and coboundary of $\sigma_\alpha^k$ and $\Delta_\alpha^{n-k}$ respectively,
then at $p' \in \sigma_j \cap \Delta_j$,

$$i_{p'}(\sigma_j^{k-1} \cdot \Delta_j^{n-k+1}) = (-1)^{n-k+1}.$$

Let $\delta(\Delta_\alpha^{n-k}) = \epsilon \sum_{j=0}^{k} (-1)^j \Delta_j^{n-k+1}$ where $\epsilon = \pm 1$.

$\Rightarrow$ We claim $\epsilon = (-1)^{n-k+1}$.

Assume $\langle \delta(\Delta_\alpha^{n-k}), \sigma_j^{k-1} \rangle = (-1)^j \langle \Delta_\alpha^{n-k}, \sigma_\alpha^k \rangle$
and we didn't give any orientation on $\Delta_j^{n-k+1}$ ~~yest~~ yet,
do not assume $\cancel{\Leftarrow}$ $i_{p'}(\sigma_j^{k-1} \cdot \Delta_j^{n-k+1}) = +1$. $\Delta_j^{n-k+1}$ is oriented
since ~~one of~~ part of $\cancel{\Delta_j}$ the boundary of $\Delta_j^{n-k+1}$ is oriented ~~by~~.
This is the same sort of argument as made in the verification of homology-invariance of intersection number.

The simplex $\sigma_\alpha^k$ intersects the cell $\Delta_j^{n-k+1}$ in an arc $r$ running from the barycenter $p = \gamma(0)$ of $\sigma_\alpha^k$ to the barycenter $p' = \gamma(1)$ of the face $\sigma_j^{k-1}$ of $\sigma_\alpha^k$.