and the vanishing

$$H^1(S, \mathcal{O}(L-p-q)) = H^1(S, \mathcal{O}(L-2p)) = 0,$$

it follows that the complete linear system of L gives an embedding $\iota_L : S \longrightarrow \mathbb{P}^N$.

☞ $\mathcal{O}_{\{p,q\}}(L|_{\{p,q\}}) = L_p \oplus L_q$

Since $\mathcal{O}(L-2p) \longrightarrow \mathcal{O}(L)$ has the image which are elements whose derivative is zero,

$0 \longrightarrow \mathcal{O}(L-2p) \longrightarrow \mathcal{O}(\underset{-p}{L}) \xrightarrow{d_p} T_p'^{*} \otimes L_p \longrightarrow 0$ is exact.

Here $T_p'^{*} \otimes L_p\ (U) = \overbrace{T_p^{*\prime} \otimes L_p}^{\cong L_p}$ if $U \ni p$

$T_p'^{*} \otimes L_p\ (U) = 0$ otherwise.

$H^1(S, \mathcal{O}(L-p-q)) = 0$ since $\deg(L-p-q) = \deg L - 2 > \deg K_S$, $\Rightarrow H^1(S, \mathcal{O}(L-p-q)) = H^1(S, \Omega^1((L-p-q) \otimes K_S^*)) = 0$. Similarly, $H^1(S, \mathcal{O}(L-2p)) = 0$.

$\Rightarrow H^0(S, \mathcal{O}(L-p-q)) \longrightarrow H^0(S, \mathcal{O}(L)) \xrightarrow{r_{p,q}} L_p \oplus L_q \to 0$

$\Rightarrow r_{p,q}$ is surjective. Similarly, we get $H^0(S, \mathcal{O}(\underset{-p}{L})) \xrightarrow{d_x} T_x'^{*} \otimes L_x \longrightarrow 0$, and so d_x is surjective. $\Rightarrow$ By P181, $H^0(S, \mathcal{J}_x(L)) \xrightarrow{d_x} T_x'^{*} \otimes L_x$ is surjective. $\Rightarrow$ By P180, ι_L is embedding.

Summarizing, we have

1. $\deg L < 0 \Rightarrow H^0(S, \mathcal{O}(L)) = 0$.
2. $\deg L > \deg K_S \Rightarrow H^1(S, \mathcal{O}(L)) = 0$
3. $\deg L > \deg K_S + 2 \Rightarrow \iota_L : S \longrightarrow \mathbb{P}^N$ is well-defined and an embedding.

☞ Choose a point $p \in S$, and consider $L = [\ell p]$, $\ell > \deg K_S + 2$.
$\Rightarrow \deg L = \ell > \deg K_S + 2 \Rightarrow \iota_L : S \longrightarrow \mathbb{P}^N$ is an embedding