$$q=0 \Rightarrow \quad \begin{array}{ccccc} H^0(X, \mathcal{A}^{p-1}) & \xrightarrow{d} & H^0(X, \mathcal{A}^p) & \xrightarrow{d} & H^0(X, \mathcal{A}^{p+1}) \\ \| & & \| & & \| \\ A^{p-1}(X) & & A^p(X) & & A^{p+1}(X) \end{array}$$

$$\Rightarrow \quad \frac{\ker d}{\operatorname{Im} d} = H^p_{DR}(X) = {}''E_2^{p,0}$$

$$\Rightarrow \quad ({}''E_{\mathcal{A}^*})_2^{p,q} = \begin{cases} H^p_{DR}(M), & q=0 \\ 0, & q>0. \end{cases}$$

Combining the previous remarks yields again the de Rham isomorphism

$$H^*(M, \mathbb{R}) \cong H^*_{DR}(M).$$

By the argument on P441 note, we get

$$H^*_{DR}(M) \cong \mathbb{H}^*(M, \mathbb{R}^*).$$

$\Rightarrow$ Since $\mathbb{H}^*(M, \mathbb{R}^*) \cong H^*(M, \mathbb{R})$, $H^*(M, \mathbb{R}) \cong H^*_{DR}(M)$.

This is, of course, essentially the previous sheaf-theoretic proof of the theorem. However, it is cast in such a way that the essential aspects are more clearly isolated, thus leading naturally to the generalizations to appear shortly.

2. Same for Dolbeault. Suppose M is a complex manifold, and let $(\mathcal{A}^{p,*}, \bar\partial)$ denote the Dolbeault complex of sheaves

$$\mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \longrightarrow \cdots,$$

and Ω^{p*} the trivial complex

$$\Omega^p \longrightarrow 0 \longrightarrow 0.$$