Plug in $x_n = 0$.

$$\Rightarrow (x_1+\cdots+x_{n-1})^n - \Big[\underbrace{\{(x_1+\cdots+x_{n-1})^n\}}_{1} + \{\underbrace{(x_2+\cdots+x_{n-1})^n+\cdots+(x_1+\cdots+x_{n-2})^n}_{{}_nC_1 - \frac{{}_nC_1\times 1}{n}}\}\Big]$$

$$+\Big[\{\overset{\#\text{ of }(\ )^n\text{'s missing two } x_i\text{'s}}{(\quad)^n \cdots}\} + \{\overset{\#\text{ of }(\ )^n\text{'s missing three } x_i\text{'s}}{\quad}\}\Big]$$

$$-\Big[\{(\quad)^n\text{'s missing three } x_i\text{'s}\} + \{(\quad)^n\text{'s missing four } x_i\text{'s}\}\Big]$$

$$\vdots$$

$$\underbrace{+(-1)^{n+1}\{(\quad)^n\text{'s missing } (n-1)\ x_i\text{'s}\}}_{\parallel}$$

$$x_1^n+\cdots+x_{n-1}^n.$$

$\Rightarrow$ Since everything is (appears equally) symmetric, we have only to check that successive { }'s cancel each other.

$\sum_{i_1<\cdots<i_k} (x_{i_1}+\cdots+x_{i_k})^n$ has ${}_nC_k$ number of ( )'s

and # of ( )^n's missing $(n-k)$ is $\frac{{}_nC_k \times (n-k)}{n}$.

$$\Rightarrow {}_nC_k - \frac{{}_nC_k\times(n-k)}{n} = \frac{{}_nC_{k-1}\times(n-k+1)}{n}$$

$$= \frac{1}{n}\,\frac{n!\,(n-k+1)}{(n-k+1)!\,(k-1)!}$$

$$= \frac{1}{n}\,\frac{n!}{(n-k)!\,(k-1)!}$$

$$= \frac{k}{n}\,\frac{n!}{(n-k)!\,k!} = \frac{k}{n}\,{}_nC_k \quad //$$