dimension $2n-1$ in $\mathbb{P}^{2n}$. $\Rightarrow$ By the assumption of the induction, since $\tilde{F}_{2n-1}$ contains an irreducible $n(n+1)/2$-dimensional family of $(n-1)$-planes, Σ'_{2n-1} is irreducible of dimension $n(n+1)/2$.

Note that $\pi_1 : I \longrightarrow F$ is onto, for, given any $p \in F$, $T_p(F) \cap F$ is the cone through p over a smooth quadric $\tilde{F}_{2n-1}$, and by the assumption of the induction $\tilde{F}_{2n-1}$ has a $(n-1)$-plane. So the n-plane spanned by p and a $(n-1)$-plane in $\tilde{F}_{2n-1}$ is what we want, i.e., $\pi_1^{-1}(p) \neq \phi$.

$\Rightarrow$ $\pi_1 : I \longrightarrow F$ is a smooth fibration $\Rightarrow$ I is irreducible of dim $2n+1 + n(n+1)/2$.

consider the choosing the generic $2n$-plane see note p810 back

Finally, since the map $\pi_2 : I \longrightarrow \Sigma'_n$ has fiber dimension n, we see that Σ'_n is irreducible of dimension

$$\frac{n(n+1)}{2} + 2n+1 - n = \frac{(n+1)(n+2)}{2},$$

and part 1 of our proposition is proved.

⊩ Since I is irreducible and $\pi_2(I) = \Sigma'_n$, Σ'_n is irreducible. $\Rightarrow$ $\frac{n(n+1)}{2} + 2n+1 = \dim \Sigma'_n + n$ dim of fiber of π_2

$$\Rightarrow \quad \dim \Sigma'_n = \frac{n(n+1)}{2} + 2n+1-n = \frac{n(n+1)}{2} + n+1 = \frac{(n+1)(n+2)}{2}.$$

Now let $F_{2n} \subset \mathbb{P}^{2n+1}$ be a smooth quadric; again, we set