$\ulcorner$ $\mathcal{L}_{(2)}$ consists of holomorphic functions on some nbd of L in $\mathbb{P}^2$ modulo I^3, and since we are considering "some sort of lifting to $\mathcal{O}^*$ from $\mathcal{O}^*/_{1+I^3}$, the "lifting" $\mathcal{L}$ is a line bundle over $\mathbb{P}^2$ having a holomorphic section which is a lift from the holomorphic section of $\mathcal{L}_{(2)}$.
$\Rightarrow$ $\mathcal{L}$ is a some positive multiple of H, where H is a hyperplane bundle over $\mathbb{P}^2$. $[nH] = \mathcal{L}$.

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}((n-2)H - H) \longrightarrow \mathcal{O}_{\mathbb{P}^2}((n\ \ H)) \longrightarrow \mathcal{O}_{L_{(2)}}(\mathcal{L}_{(2)}) \longrightarrow 0$$

since, given $\sigma \in \mathcal{O}_{\mathbb{P}^2}(nH)$ s.t $\sigma = 0$ in $\mathcal{O}_{L_{(2)}}(\mathcal{L}_{(2)})$, then σ vanishes up to order 3 locally, and so $\exists$ $\tau \in \mathcal{O}_{\mathbb{P}^2}((n-3)H)$ s.t $S_0 \otimes \tau = \sigma$ locally, at stalk, where $(S_0 = 0) = 3H$.

$$\Rightarrow \quad 0 \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}((n-3)H)) \longrightarrow H^0(\mathbb{P}^2, \mathcal{O}(nH)) \longrightarrow$$

$$H^0(L_{(2)}, \mathcal{L}_{(2)}) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}((n-3)H)) = 0$$

$$\parallel$$

$$H^0(\mathbb{P}^2, \mathcal{O}(-nH)) = 0 \quad \text{since } 0+0<2$$

by P155 & P156.

$$\Rightarrow \quad 0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}((n-3)H)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(nH)) \longrightarrow H^0(\mathcal{O}(\mathcal{L}_{(2)})) \longrightarrow 0.$$

Combining this with the previous paragraph, we conclude: There is exactly one condition that the second-order arc C_i $(i = 1, 2, \cdots n)$ be cut out by an algebraic curve C in $\mathbb{P}^2$. The Reiss relation gives this