Thus, for a harmonic function φ,

$$\int_{\|x\|=\delta} \varphi\sigma = \int_{\|x\|=\epsilon} \varphi\sigma.$$

「Since the left of (*) is zero, we get the result.」

If we let $\delta \to 0$, the left-hand side tends to $\varphi(0)$, and the mean-value property is established.

Now we assume that $\chi(x)$ is radially symmetric. Then a harmonic function φ satisfies $\varphi_\epsilon = \varphi$ for $\epsilon > 0$.

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$$\varphi_\epsilon(x) = \int_{\mathbb{R}^n} \varphi(y)\,\chi_\epsilon(x-y)\,dy$$

For fixed $x \in \mathbb{R}^n$, $\displaystyle\int_{\mathbb{R}^n} \varphi(y)\,\chi_\epsilon(x-y)\,dy$

$$= \int_0^\infty \int_{\|y-x\|=r} \varphi(y)\,\chi_\epsilon(x-y)\,\sigma_y(x)\,\frac{C_n}{r^{n-1}}\,dr$$

$$= \int_0^\infty \int_{\|y-x\|=r} \varphi(y)\,\chi_\epsilon(r)\,\sigma_y(x)\,\frac{C_n}{r^{n-1}}\,dr$$ (since χ_ϵ is radially symmetric)

$$= \int_0^\infty \chi_\epsilon(r)\left(\frac{C_n}{r^{n-1}}\right)^{-1} \int_{\|y-x\|=r} \varphi(y)\,\sigma_y(x)\,dr$$

$$= \int_0^\infty \chi_\epsilon(r)\left(\frac{C_n}{r^{n-1}}\right)^{-1}\varphi(x)\,dr = \varphi(x)\int_0^\infty \chi_\epsilon(r)\left(\frac{C_n}{r^{n-1}}\right)^{-1} dr$$