Since f is transversal to Z,

$$D(\psi\circ f\circ\varphi^{-1})(x_0) + (n-k) = n.$$

$\Rightarrow$ This implies that $U \xrightarrow{\psi\circ f\circ\varphi^{-1}} V \xrightarrow{\pi_k} R^k$ is a submersion where $\pi_k : R^n \longrightarrow R^k \times \{0\}$. is a natural projection.

$$\Rightarrow \operatorname{rank}\left(\frac{\partial \psi_i\circ f}{\partial \varphi_j}\right)(x_0)_{\substack{1\le i\le m\\ 1\le j\le k}} \text{ is } k.$$

$$\pi_k \circ \left(\frac{\partial \psi_i\circ f}{\partial \varphi_j}\right)(x_0) = \left(\frac{\partial \psi_i\circ f}{\partial \varphi_j}\right)_{\substack{1\le i\le m\\ 1\le j\le k}}.$$

Let's assume that $\left(\frac{\partial \psi_i\circ f}{\partial \varphi_j}\right)_{\substack{1\le i\le k\\ 1\le j\le k}}$ is a non-singular $k\times k$ matrix.

$\Rightarrow$ Define a map $F : U \longrightarrow R^k \times R^{m-k}$

$$(x_1, x_2, \cdots x_m) \longmapsto (\psi_1\circ f\circ\varphi^{-1}, \cdots \psi_k\circ f\circ\varphi^{-1}, x_{k+1}, \cdots x_m)$$

$\Rightarrow$ $DF(x_0)$ is again non-singular.

$\Rightarrow$ By the inverse function theorem, F is a diffeomorphism from $\underset{R^m}{\overset{U_0}{\cap}}$ to $\underset{R^m}{\overset{V_0}{\cap}}$ s.t.

$$U_0 \subset U \qquad V_0 \subset R^k \times R^{m-k}.$$

Restrict F to $(\psi\circ f\circ\varphi^{-1})(y_0)$, and then

$$F| : (\psi\circ f\circ\varphi^{-1})(y_0) \longrightarrow ((y_0)_1, \cdots (y_0)_k, x_{k+1}, \cdots x_m)$$

is diffeomorphic. $\Rightarrow$ $f^{-1}(Z)$ is a submanifold of dim $m-k$.