Then for a generic point $p = [\alpha_0, \alpha_1, \cdots \alpha_k]$ in $\mathbb{P}^k$, the inverse image of p in $\pi'(V) \subset \mathbb{P}^{k+1}$ would consist of at most of the d' points $\{[\alpha_0, \cdots \alpha_k, \beta]\}$,

where
$$\beta^{d'} + \psi_1\left(\frac{\alpha_1}{\alpha_0}, \cdots \frac{\alpha_k}{\alpha_0}\right)\beta^{d'-1} + \cdots + \psi_{d'}\left(\frac{\alpha_1}{\alpha_0}, \cdots \frac{\alpha_k}{\alpha_0}\right) = 0.$$

$\ulcorner$ We don't need terms of $x_{k+2}, \cdots x_n$, since $x_1, x_2 \cdots x_n$ are independent variables. $x_{k+1} \in K(V)$.

For a generic point $p = [\alpha_0, \alpha_1 \cdots \alpha_k]$ where $\alpha_0 \neq 0$, since
$$x_{k+1}^{d'} + \psi_1(x_1 \cdots x_k) \cdot x_{k+1}^{d'-1} + \cdots + \psi_{d'}(x_1, \cdots x_k) \equiv 0,$$
$d' < d$,
$$\left(\frac{\beta}{\alpha_0}\right)^{d'} + \psi_1\left(\frac{\alpha_1}{\alpha_0}, \cdots \frac{\alpha_k}{\alpha_0}\right)\left(\frac{\beta}{\alpha_0}\right)^{d'-1} + \cdots + \psi_{d'}\left(\frac{\alpha_1}{\alpha_0}, \cdots \frac{\alpha_k}{\alpha_0}\right) = 0 \quad \circledast$$

$\Rightarrow \exists$ at most of d' points $\{[\alpha_0, \cdots \alpha_k, \beta]\}$ satisfying the equation $\circledast$. $\lrcorner$

But since the projection $\pi' : V \longrightarrow \mathbb{P}^{k+1}$ is generically one to one onto its image and the fibers of π generically consist of $d > d'$ points, this is impossible. Q.E.D.

$\ulcorner$

$$\begin{tikzcd} U \subset V \arrow[rr, "\pi'"] \arrow[dr, "\pi"'] & & \mathbb{P}^{k+1} \arrow[dl] \\ & \mathbb{P}^k & \end{tikzcd}$$

Choose a generic point $p = [\alpha_0, \alpha_1, \cdots \alpha_k]$ so that $\pi^{-1}(p) \subset U$, where U is the open set above. This is possible.

$[\alpha_0, \alpha_1, \cdots \alpha_k] = p \in$