$q \geq 1.$ $\quad H^q(\underline{U}, \mathcal{O}) = H_{\bar\partial}^{0,q}(M) = H^q(M, \mathcal{O}).$

Dolbeault Theorem.

$q = 0.$ $\quad H^0(\underline{U}, \mathcal{O}) = H^0(M, \mathcal{O})$ $\quad$ done. !

The same argument works as well for the sheaves Ω^P. i.e.

$H^q(\underline{U}, \Omega^P) = H^q(M, \Omega^P)$. if $\underline{U} = \{U_\alpha\}$ is acyclic for the sheaf Ω^P.

$$H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}, \Omega^P) = 0. \quad q > 0, \text{ for all } r.$$

pf). $\quad H^q(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}, \Omega^P) = H_{\bar\partial}^{P,q}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}) = 0$

$\Leftrightarrow \quad Z_{\bar\partial}^{P,q}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}) = \bar\partial\, \mathcal{A}^{P,q-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r})$

Throw this away.

Let $r = p$, $q = r$

$\Rightarrow$ We have exact sequences

$$0 \to C^P(\underline{U}, Z_{\bar\partial}^{P,\overset{r}{\overset{\shortparallel}{q}}-1}) \longrightarrow C^P(\underline{U}, \mathcal{A}^{P,\overset{r}{\overset{\shortparallel}{q}}-1}) \xrightarrow{\bar\partial} C^P(\underline{U}, Z_{\bar\partial}^{P,\overset{r}{\overset{\shortparallel}{q}}}) \to 0$$

$\Rightarrow \quad H^P(\underline{U}, Z_{\bar\partial}^{P,r-1}) \longrightarrow H^P(\underline{U}, \mathcal{A}^{P,r-1}) \longrightarrow H^P($

From the sequence $\quad 0 \to Z_{\bar\partial}^{P,q-1} \longrightarrow \mathcal{A}^{P,q-1} \longrightarrow Z_{\bar\partial}^{P,q} \to 0,$

and $\quad Z_{\bar\partial}^{P,q}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}) = \bar\partial\, \mathcal{A}^{P,q-1}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_r}),$

we have exact sequences (not in a sense of sheaf but "exactly" sequence)

$$0 \to C^r(\underline{U}, Z_{\bar\partial}^{P,q-1}) \longrightarrow C^r(\underline{U}, \mathcal{A}^{P,q-1}) \longrightarrow C^r(\underline{U}, Z_{\bar\partial}^{P,q}) \to 0$$

So that we get another sequence

$$\begin{array}{cccccc} H^r(\underline{U}, Z_{\bar\partial}^{P,q-1}) \longrightarrow & H^r(\underline{U}, \mathcal{A}^{P,q-1}) & \longrightarrow & H^r(\underline{U}, Z_{\bar\partial}^{P,q}) \longrightarrow & H^{r+1}(\underline{U}, Z_{\bar\partial}^{P,q-1}) \\ & \shortparallel & & & \downarrow \\ & 0 & & & 0 = H^{r+1}(\underline{U}, \mathcal{A}^{P,q-1}) \\ & \text{for } r > 0 & & & \end{array}$$