In particular,

$$\left[\frac{i}{2\pi}\Theta\right] = \left[\frac{i}{2\pi}\Theta'\right].$$

⊢ $\partial\bar\partial\rho = -\frac{1}{2}dd^*\rho = d(-\frac{1}{2}d^*\rho) = 0$ in $H^2_{DR}(M)$ ⊣

Working in the other direction, suppose that $\frac{i}{2\pi}\varphi$ is a real, closed $(1,1)$-form representing $C_1(L)$ in $H^2_{DR}(M)$. If we can solve the equation

$$\Theta = \partial\bar\partial\rho + \varphi$$

for a real C^∞ function ρ, then the metric $e^\rho|s|^2$ on L will have curvature form φ.

⊢ If we solve $\Theta = \partial\bar\partial\rho + \varphi$, where Θ is the curvature form w.r.t. $|s|^2$,

$|s|'^2 = e^\rho|s|^2$ gives the curvature Θ' satisfying the following

$$\Theta = \partial\bar\partial\rho + \Theta'$$

$$\Rightarrow \quad \partial\bar\partial\rho + \Theta' = \partial\bar\partial\rho + \varphi \quad \Rightarrow \quad \Theta' = \varphi. \quad ⊣$$

Our proposition therefore follows from the

Lemma. If η is any (p,q)-form on a compact Kähler manifold, and η is d, ∂- or $\bar\partial$-exact, then

$$\eta = \partial\bar\partial\gamma$$

for some $(p-1,q-1)$-form γ. If $p=q$ and η is real, then we may take $i\gamma$ also to be real.