The Poincaré-dualized version of the hard Lefschetz theorem says that the operation of intersection with an $(N-k)$-plane $\mathbb{P}^{N-k} \subset \mathbb{P}^N$ gives an isomorphism

$$H_{n+k}(M) \xrightarrow{\cap \mathbb{P}^{N-k}} H_{n-k}(M).$$

$\ulcorner$ First, note that, if $\sigma \in H_k(M)$, $\tau \in H_\ell(M)$, and η_σ, η_τ are Poincare-dual forms of $H^{2n-k}(M)$ and $H^{2n-\ell}(M)$, the Poincare-dual of $\sigma \cap \tau$ is $\eta_\sigma \wedge \eta_\tau \in H^{4n-(k+\ell)}(M)$.

pf) What we have to show is that for every $\alpha \in H_{-(k+\ell)+4n}(M)$,

$$\int_\alpha \eta_\sigma \wedge \eta_\tau = \#((\sigma \cap \tau) \cdot \alpha).$$

For each fixed $\alpha \in H_{-(k+\ell)+4n}(M)$,

$$\int_{\beta \cap \alpha} \iota^* \eta_\sigma = \int_\beta \eta_\sigma = \#(\sigma \cdot \beta) = \#((\sigma \cap \alpha) \cdot (\beta \cap \alpha))$$

$$\begin{array}{ccc} \alpha & \overset{\iota}{\hookrightarrow} & M \\ & & \cup \\ & & \beta \end{array}$$

$\Rightarrow$ $\iota^* \eta_\sigma$ is Poincare-dual to $\sigma \cap \alpha$ in α

Similarly $\iota^* \eta_\tau$ is Poincare-dual to $\tau \cap \alpha$ in α

$\Rightarrow$ By P59, $\iota^* \eta_\sigma \wedge \iota^* \eta_\tau$ is Poincare-dual to $(\sigma \cap \alpha) \cap (\tau \cap \alpha) = (\sigma \cap \tau) \cap \alpha$.

In other words,

$$\int_{\gamma \cap \alpha} \iota^* \eta_\sigma \wedge \iota^* \eta_\tau = \int_\gamma \eta_\sigma \wedge \eta_\tau = \#(((\sigma \cap \tau) \cap \alpha) \cdot \gamma)$$

$$= \#((\sigma \cap \tau) \cdot \gamma) \quad \lrcorner$$