Lemma. If M is an algebraic surface with the same Betti numbers as $\mathbb{P}^2$ and K_M is not positive, then $M \cong \mathbb{P}^2$.

Proof. Since $b_1(M) = 0$, we have $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z}) \cong \mathbb{Z}$.

⌈ See P139. From $0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$,

$$\begin{array}{ccccccccc} H^1(M,\mathbb{Z}) & \to & H^1(M,\mathcal{O}) & \to & H^1(M,\mathcal{O}^*) & \to & H^2(M,\mathbb{Z}) & \to & H^2(M,\mathcal{O}) \\ \| & & \| & & \| & & & & \\ 0 & & 0 & & \mathrm{Pic}(M) & & & & \end{array}$$

Since $H^1(M,\mathcal{O}) = H^{0,1}(M)$ and $H^1(M) = H^{1,0}(M) \oplus H^{0,1}(M)$,

$H^1(M) = 0 \Rightarrow H^{1,0}(M) = H^1(M,\mathcal{O}) = 0$.

$\Rightarrow H^1(M, \mathcal{O}^*) \hookrightarrow H^2(M, \mathbb{Z})$ is injective.

$H^2(M,\mathcal{O}) = H^{0,2}(M)$.

$H^2(M) = H^{2,0}(M) \oplus H^{0,2}(M) \oplus H^{1,1}(M)$.

If $H^{0,2}(M) \neq 0$, then $\dim H^2(M) = b^2(M) \geq 2$ which contradicts to $b^2(M) = b^2(\mathbb{P}^2) = 1$. (since $\dim H^{2,0}(M) = \dim H^{0,2}(M)$)

$\Rightarrow H^2(M,\mathcal{O}) = 0. \Rightarrow H^1(M,\mathcal{O}^*) \cong H^2(M,\mathbb{Z}) \cong \mathbb{Z}$. ⌋

Since M is algebraic, there exists a positive line bundle L' on M; let L be the generator of $\mathrm{Pic}(M)$ s.t. $L' = L^n$ for some $n > 0$.

⌈ $M \hookrightarrow \mathbb{P}^N \Rightarrow$ For a generic hyperplane H,

$$\begin{tikzcd} i^*[H] \arrow[r] \arrow[d] & {[H]} \arrow[d] \\ M \arrow[r, hook, "i"] & \mathbb{P}^N \end{tikzcd} \Rightarrow \underset{= L'}{i^*[H]} \text{ is positive. See P150.}$$ ⌋