$\otimes e_\beta$

$$= \sum [\Lambda, \bar\partial](\eta_\alpha) \otimes e_\alpha + \sum \Lambda(\eta_\alpha \wedge \theta''_{\alpha\beta}) \otimes e_\beta - \sum \Lambda(\eta_\alpha) \wedge \theta''_{\alpha\beta} \otimes e_\beta$$

$$= \sum [\Lambda, \bar\partial](\eta_\alpha) \otimes e_\alpha + [\Lambda, \theta''] \eta.$$

where $\theta''(\eta) = \theta''(\sum \eta_\alpha \otimes e_\alpha) = \sum \eta_\alpha \wedge \theta''_{\alpha\beta} \otimes e_\beta$

$$\Rightarrow \quad \sum \Lambda(\eta_\alpha) \wedge \theta''_{\alpha\beta} \otimes e_\beta = \theta''(\sum \Lambda(\eta_\alpha) \otimes e_\alpha) = \theta''(\Lambda(\eta))$$

"$\theta''\Lambda(\eta)$"

$$\sum \Lambda(\eta_\alpha \wedge \theta''_{\alpha\beta}) \otimes e_\beta = \Lambda(\sum \eta_\alpha \wedge \theta''_{\alpha\beta} \otimes e_\beta)$$
$$= \Lambda(\theta''(\eta)) = \Lambda \theta''(\eta)$$

$$* \quad \langle L(\tau \otimes s), \eta \otimes e \rangle = \langle \tau \otimes s, \Lambda(\eta \otimes e) \rangle$$
$$= \langle \omega \wedge \tau \otimes s, \eta \otimes e \rangle = \langle \omega \wedge \tau, \eta \rangle \langle s, e \rangle$$
$$= \langle L(\tau), \eta \rangle \langle s, e \rangle = \langle \tau, \Lambda(\eta) \rangle \langle s, e \rangle$$
$$= \langle \tau \otimes s, \Lambda(\eta)\ e \rangle = \langle \tau \otimes s, \Lambda(\eta \otimes e) \rangle$$

Thus $\Lambda(\sum \eta_\alpha \otimes e_\alpha) = \sum \Lambda(\eta_\alpha) \otimes e_\alpha$ $\quad *$

$$\Rightarrow [\Lambda, \bar\partial] \eta = \sum [\Lambda, \bar\partial](\eta_\alpha) \otimes e_\alpha + [\Lambda, \theta''] \eta$$
$$= \sum -i\partial^*(\eta_\alpha) \otimes e_\alpha + [\Lambda, \theta''] \eta.$$

Similarly,

$$D'\eta = \sum_\alpha \partial \eta_\alpha \otimes e_\alpha + \sum_{\alpha, \beta} (\eta_\alpha \wedge \theta'_{\alpha\beta}) \otimes e_\beta$$

i.e

$$D'^* \eta = \sum \partial^* \eta_\alpha \otimes e_\alpha + \theta'^* \eta.$$

$$D'\eta = \sum_\alpha \partial \eta_\alpha \otimes e_\alpha + \sum_{\alpha, \beta} (\eta_\alpha \wedge \theta'_{\alpha\beta}) \otimes e_\beta$$
$$= \sum \partial \eta_\alpha \otimes e_\alpha + \theta'(\eta)$$

$$\langle \partial \eta_\alpha \otimes e_\alpha, \tau \otimes s \rangle = \langle \partial \eta_\alpha, \tau \rangle \langle e_\alpha, s \rangle = \langle \eta_\alpha, \partial^* \tau \rangle \langle e_\alpha, s \rangle$$
$$= \langle \eta_\alpha \otimes e_\alpha, \partial^* \tau \otimes s \rangle$$