$E \otimes L^k$ are Poincare dual to the intersection of M with suitable Schubert cycles. In summary, the Chern classes of a holomorphic vector bundle over an algebraic variety are represented by fundamental classes of algebraic cycles.

$\ulcorner$ By Gauss-Bonnet Formula II on P413, $C_p(E\otimes L^k)$ is Poincare dual to the degeneracy cycle D_{r-p+1}. $\Rightarrow$ By the argument on P412, $\iota^{-1}(\sigma_{\underbrace{1,\dots,1}_{p}}(V)) = D_{r-p+1}$.

$$\begin{array}{ccccc}
H^{2p}(M) & \longleftrightarrow & H_*(M) & \xrightarrow{\iota_*} & H_*(G(r,N)) \quad \iota \text{ is embedding} \\
\Downarrow & & \Downarrow & & \Downarrow \\
C_p(E\otimes L^k) & \longleftrightarrow & [D_{r-p+1}] & \longleftrightarrow & [\overbrace{\iota(M)}^{\cong M} \cap \sigma_{1,\dots,1}(V)] \\
& & & & \uparrow \\
& & & & \text{algebraic cycles.}
\end{array}$$

$\Rightarrow$ Since the Chern classes of $E\otimes L^k$ are represented by algebraic cycles and $C(L^{-k}) = 1-k\eta_D$, by the fact that the Chern classes of $E \cong (E\otimes L^k)\otimes L^{-k}$ can be expressed as polynomials in the Chern classes of $E\otimes L^k$ and L^{-k}, the Chern classes of E are represented by fundamental classes of algebraic cycles. $\lrcorner$

There is also a notion of positivity for the Chern classes of holomorphic vector bundles. We shall not enter into this in detail, as it will not be used in the study of specific varieties, but will offer two observations. If $E \to M$ is generated by its global holomorphic sections, we have seen at the end of Section 5 in Chapter 0 that there is a hermitian connection whose curvature matrix has the local form