Suppose that for some $x_0 \in U$, every $k \times k$ minor matrix of $(a_{ij}(x_0))$ has rank less than k.
$\Rightarrow$ $(a_{ij}(x_0))$ has rank less than k.
It is impossible. For, since $\{\sigma_1(x_0) \cdots \sigma_n(x_0)\}$ spans E_{x_0}, let's assume that $\langle \sigma_1(x_0), \cdots \sigma_k(x_0) \rangle = E_{x_0}$.

$$\Rightarrow \quad \begin{aligned} \sigma_1(x_0) &= a_{11} e_1 + \cdots + a_{k1} e_k \\ \sigma_2(x_0) &= a_{12} e_1 + \cdots + a_{k2} e_k \\ &\vdots \\ \sigma_k(x_0) &= a_{1k} e_1 + \cdots + a_{kk} e_k \end{aligned}$$

$\Rightarrow$ Since $\{e_1(x_0) \cdots e_k(x_0)\}$ is a basis for E_{x_0},

$$\begin{pmatrix} a_{11}(x_0) & \cdots & a_{k1}(x_0) \\ \vdots & & \\ \vdots & & \\ a_{1k}(x_0) & \cdots & a_{kk}(x_0) \end{pmatrix}$$ must be invertible.

$$\Rightarrow \quad \operatorname{rank}\left(a_{ij}(x_0)\right)_{1 \le i,j \le k} = k.$$

$\Rightarrow$ $e_1, \cdots e_k$ can be expressed as a linear combination of $\sigma_1, \sigma_2 \cdots \sigma_k$ for $\forall x \in U'$, i.e U', open, contains x_0.

$$\Rightarrow \quad \begin{aligned} e_1 &= b_{11}(x)\sigma_1 + \cdots + b_{k1}(x)\sigma_k \\ &\vdots \\ e_k &= b_{1k}(x)\sigma_1 + \cdots + b_{kk}(x)\sigma_k . \end{aligned}$$

$$\Rightarrow \quad \sigma = c_1 e_1 + \cdots + c_k e_k \qquad \text{over } U' \subset U$$
$$= c_1(b_{11}\sigma_1 + \cdots + b_{k1}\sigma_k)$$
$$+ c_2(b_{12}\sigma_1 + \cdots + b_{k2}\sigma_k) + \cdots + c_k(b_{1k}\sigma_1 + \cdots + b_{kk}\sigma_k)$$
$$\in V \quad \text{locally}.$$