Now, if two of the points $p_3, \cdots p_7$ lie on the line $L = L_{12}$ — say p_3 and p_4 — we are done: the cubic $L_{25} + L_{36} + L_{47}$ contains p_1, and so either p_5, p_6, or p_7 lies on L, giving us five collinear points.

⌈ Since $L_{25} + L_{36} + L_{47}$ is a cubic passing $p_2, \cdots p_7$, $L_{25} + L_{36} + L_{47}$ contains p_1 $\Rightarrow$ $p_1 \in L_{25}$ or L_{36} or L_{47}. If $p_1 \in L_{25}$, then $\overline{p_2 p_5} \ni p_1$, and the proper transform of L_{25} contains p_1. $\Rightarrow$ $L = L_{12} \ni p_5$. $\Rightarrow$ $L \ni p_1, p_2, p_3, p_4, p_5$. Similarly for L_{36} and L_{47}. $\Rightarrow$ We have five collinear points. Here $L = L_{12}$ is the tangent line of C at p_2 with the slope p_1. ⌟

If exactly one of the points $p_3, \cdots, p_7$ — say p_3 — lies on L, then the cubics

$$L_{24} + L_{35} + L_{67}, \quad L_{24} + L_{36} + L_{57}, \quad \text{and} \quad L_{25} + L_{36} + L_{47}$$

all contain p_1, and so must be singular at p_2; thus p_2 lies on L_{47}, L_{57}, and L_{67}, i.e., p_4, p_5, and p_6 all lie on the line L_{27}.

⌈ ① $p_1 \in L_{24} + L_{35} + L_{67}$

$L_{12} \ni p_4$ ↑⇓

(i) $p_1 \in L_{24}$ $\Rightarrow$ Obviously, $L_{24} = L_{23} \Rightarrow *$, since L_{12} contains only p_3

(ii) $p_1 \in L_{35}$ $\Rightarrow$ L_{35} must contain p_2 $\Rightarrow$ $L_{35} = L_{23} \Rightarrow *$ ⇑

(iii) $p_1 \in L_{67}$ $\Rightarrow$ L_{67} 〃 p_2 $\Rightarrow$ $L_{67} = L_{23} \Rightarrow *$

$\Rightarrow$ $L_{24} + L_{35} + L_{67}$ is singular at p_2, and $p_2 \in L_{67}$.

② $p_1 \in L_{24} + L_{36} + L_{57}$.