$$f'(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^2} = \frac{\cos\theta - (\sin\theta)/\theta}{\theta}$$
$$= \frac{\cos\theta - \cos\xi}{\theta} \quad (0 < \xi < \theta).$$

Since the cosine is a decreasing function in the interval $[0, \pi/2]$, $f'(\theta) < 0$ and the result follows. ■

Example 9.33. We wish to show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Our first inclination is to integrate $(\sin z)/z$ along the same contour as in the previous example. This does not work for two reasons. First, $(\sin z)/z$ has a singularity at $z = 0$ and we can not usually integrate along a path that passes through a singularity point. But the singularity is removable; so this difficulty can be overcome. Second, and more important as was indicated earlier, the integral of $(\sin z)/z$ along the semicircle does not approach a finite limit as the radius tends to infinity, because for $z = iR$ one sees that

$$\lim_{R\to\infty} \frac{\sin(iR)}{iR} = \lim_{R\to\infty} \frac{e^{-R} - e^{R}}{2i^2 R} \to \infty \quad \text{as } R \to \infty.$$

We will consider the function e^{iz}/z, whose imaginary part on the real axis is $(\sin x)/x$. Our contour C will consist of the real axis from ϵ to R, the semicircle in the upper half-plane from R to $-R$, the real axis from $-R$ to $-\epsilon$, and the semicircle in the upper half-plane from $-\epsilon$ to ϵ (see Figure 9.5). The function e^{iz}/z is analytic inside and on C, so that

$$\begin{aligned} 0 &= \int_C \frac{e^{iz}}{z}\, dz \\ &= \int_\epsilon^R \frac{e^{ix}}{x}\, dx + \int_0^\pi \frac{e^{iRe^{i\theta}}}{Re^{i\theta}} iRe^{i\theta}\, d\theta + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}\, dx + \int_\pi^0 \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\, d\theta \\ &= \int_\epsilon^R \frac{e^{ix} - e^{-ix}}{x}\, dx + i\int_0^\pi e^{iRe^{i\theta}}\, d\theta - i\int_0^\pi e^{i\epsilon e^{i\theta}}\, d\theta \end{aligned}$$

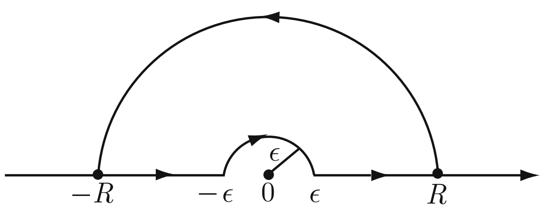

Figure 9.5.

where we have replaced x by $-x$ in the third integral and combined with the first integral. Since $e^{ix} - e^{-ix} = 2i \sin x$, the last equation may be rewritten as

$$0 = 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\, dx + i \int_0^{\pi} e^{iRe^{i\theta}}\, d\theta - i \int_0^{\pi} e^{i\epsilon e^{i\theta}}\, d\theta. \tag{9.21}$$

We now examine the behavior of the second integral on the left side of (9.21). From the identity $\sin(\pi - \theta) = \sin\theta$ and the lemma, it follows that

$$\begin{aligned} \left| i \int_0^{\pi} e^{iRe^{i\theta}}\, d\theta \right| &\leq \int_0^{\pi} e^{-R\sin\theta}\, d\theta = 2 \int_0^{\pi/2} e^{-R\sin\theta}\, d\theta \\ &\leq 2 \int_0^{\pi/2} e^{-(2R/\pi)\theta}\, d\theta \\ &= \frac{\pi}{R}(1 - e^{-R}), \end{aligned}$$

which tends to 0 as R approaches ∞. Hence letting $R \to \infty$ in (9.21) leads to

$$2 \int_{\epsilon}^{\infty} \frac{\sin x}{x}\, dx = \int_0^{\pi} e^{i\epsilon e^{i\theta}}\, d\theta. \tag{9.22}$$

For $0 < \epsilon < 1/2$, we expand $e^{i\epsilon e^{i\theta}}$ in a power series to show that

$$|e^{i\epsilon e^{i\theta}} - 1| < 2\epsilon$$

for all θ, $0 < \theta \leq \pi$. We see that

$$\int_0^{\pi} e^{i\epsilon e^{i\theta}}\, d\theta = \int_0^{\pi} (e^{i\epsilon e^{i\theta}} - 1)\, d\theta + \int_0^{\pi} d\theta \to \pi \quad \text{as } \epsilon \to 0.$$

Thus, letting $\epsilon \to 0$ in (9.22), it follows that

$$2 \int_0^{\infty} \frac{\sin x}{x}\, dx = \pi$$

and the result follows. The reader should verify that the contour in Figure 9.6 could also have been used to prove the desired result. ●

Let us demonstrate the method by evaluating another integral

$$I = \int_0^{\infty} \frac{x \sin(ax)}{x^2 + m^2}\, dx \quad (a, m > 0).$$

Note that the limits of integration in the given integral are not from $-\infty$ to ∞ as required by the method described above. On the other hand, since the integrand is an even function of x,