Philosophical Fragments

by Bernhard Riemann

Introduction to the Second German Edition

The philosophical speculations whose results—in so far as they can be assembled from his literary remains—are here communicated, concerned Riemann throughout the greater part of his life. Anything definite concerning the time at which these individual fragments were written can hardly be determined. The drafts here are far from being coherent essays ready for publication, even if many passages indicate that Riemann had at certain times intended such a publication; they suffice, in any case, to characterize Riemann's orientation to questions of psychology and natural philosophy in general and to indicate the course taken by his investigations; unfortunately, however, almost every exposition is lacking in detail. The value that Riemann himself placed on these labors can be seen from the following note:

"The tasks that principally concern me now are:

"1. To introduce the imaginary into the theory of other transcendental functions, in a manner similar to the way this has already been done with such great success for algebraic functions, the exponential and cyclical functions, and the elliptical and Abelian functions. To that end, I have supplied the most necessary general preparations in my inaugural dissertation. (See article 20 of this dissertation.)

"2. In connection with this, new methods exist for integration of partial differential equations, which I have already applied to several physical subjects with success.

"3. My principal task concerns a new conception of known natural laws—the expression of these laws by means of other fundamental concepts—through which it becomes possible to use experimental data on the reciprocal action of heat, light, magnetism, and electricity in order to investigate their relations. I was led to this principally through the study of Newton's, Euler's, and—on the other hand—Herbart's works. Concerning the latter, I could concur almost completely with Herbart's earliest investigations, whose results are expressed in his graduation and habilitation theses (of Oct. 22 and 23, 1802), but I had to diverge from the later course of his speculation on an essential point. I differ with him in regard to natural philosophy and those propositions in psychology which concern their connection to natural philosophy."

Further along, in another place, we find a more exact description of this standpoint:

"The author is a Herbartian in psychology and epistemology (methodology and the theory of perception); he cannot, however, for the most part, agree with Herbart's natural philosophy and the related disciplines (ontology and the study of continua)."

The three fragments unified under the common title "III. Natural Philosophy" have been rearranged in this second edition. Number 2 of the first edition has been exchanged with number 3. According to a conjecture of Dr. Isenkrahe in Bonn which is well supported by internal evidence, it is the essay titled "Gravitation and Light" which is referred to in the passage of Riemann's letter of Dec. 28, 1853, that is cited in the biographical sketch [pp. 539-558 of his *Collected Works*], according to which Riemann had in view a publication of these investigations. The essay, "New Mathematical Principles of Natural Philosophy," with the observation, "Discovered on March 1, 1853," which is concerned with an entirely different set of ideas, is therefore of an earlier origin, and the bold hypothesis expressed in that essay of the disappearance of matter was not further pursued by Riemann.

—Heinrich Weber (1892)

Translator's Note

This is the first English translation of various sketches left by Riemann at his death in 1866. They were compiled under the title *Fragmente philosophischen Inhalts* (Philosophical Fragments), and first appeared in the 1876 first edition of *Bemhard Riemann's Gesammelte Mathematische Werke und Wissenschaftlicher Nachlass* (Bernhard Riemann's Collected Mathematical Works and Scientific Remains), published by B.G. Teubner. The volume was edited by Heinrich Weber, who later compiled and published *Partial Differential Equations in Mathematical Physics from Riemann's Lectures*.

Teubner published a more complete second edition of Riemann's collected works in 1892, also prepared by Weber, and a supplement of additional materials (*Nachträge*) appeared separately in 1902, edited by M. Noether and W. Wirtinger. These two volumes were later reprinted by various publishers as one. Dover Publications (New York) issued such a reprint in 1953, with the title *The Collected Works of Bernhard Riemann*, although the only English content was a brief new introduction by Hans Lewy on Riemann's career and thought.

In the German edition of the fragments translated here, the individual pieces are apparently separated by the short, centered rules that have been carried over in this translation. All emphases and ellipses are in the original. Words or phrases in square brackets have been supplied by the translator. Riemann's own footnotes are indicated by asterisks and daggers, while the translator's notes are numbered and appear at the end.

The translation owes its inspiration to Lyndon H. LaRouche, and was done under the supervision of Carol White. Thanks go to William F. Wertz, Jr. and Renée Sigerson for their abundant help.

—David Cherry

I. On Psychology and Metaphysics

Do not scornfully reject the gifts I have devotedly
marshalled for you, before you have understood them.
—Lucretius

Bernhard Riemann (1826-1866)

With each simple act of thinking, something durable, substantial, enters our mind. This substance appears to us, in fact, as a unity, but it appears (insofar as it is the expression of space and time extension) as comprising a subsumed manifold; I name this a "thought mass."[1] To this effect, all thinking is the development of new thought masses.

The thought masses entering into the mind appear to us to be images; their varying internal states determine how they differ qualitatively.

As they are forming, the thought masses blend; or are folded together, or connect to one another and also to older thought masses, in a precisely determined manner. The character and strength of these connections depend upon causes which were only partially recognized by Herbart, but which I shall fill out in what follows. They rest primarily on the internal relationships among the thought masses.

The mind is a compact, multiply connected thought mass with internal connections of the most intimate kind. It grows continuously as new thought masses enter it, and this is the means by which it continues to develop.

Thought masses once formed, are imperishable; and their connections cannot be dissolved; only the relative strength of these connections is altered by the addition of new thought masses.

Thought masses need no material carrier for their continued existence, and exert no lasting effect upon the physical world. Therefore they are not related to any portion of matter, and have no position in space.

On the other hand, a material carrier is required for every entry, generation, every formation of new thought masses, and for their unification. Thus all thinking does occur at a definite place.

(It is not the retention of our experience but only thinking, which is strenuous; and this exertion of effort, in so far as we can estimate it, is proportional to the mental activity.)

Every thought mass which enters the mind, stimulates all thought mass to which it is related, and does so the more strongly the less the dissimilarity between the internal states (quality).

This stimulation is not confined, however, merely to related thought masses, but also extends, through mediation, to those that are linked with them (that is, connected by previous thought processes). Thus if among the related thought masses, a portion is linked, these will be stimulated not merely directly but also through mediation, and therefore will be stimulated proportionally more strongly than the rest.

The reciprocal action of two thought masses being formed at the same time, is conditioned by a material process between the places where they are both being formed. Likewise, for material reasons, all thought masses being formed enter into unmediated interaction with those formed immediately before; however, through mediation, all older thought masses linked to these will also be stimulated into activity, although to a weaker degree according to the diminished amount and increased distance of their connections.

The most general and simplest expression of the effectiveness of older thought masses is in their reproduction, which occurs when an active thought mass strives to reproduce one similar to itself.

The formation of new thought masses is based partly on the combined effect of older thought masses, partly on material causes; and these, working together, are retarded or advanced according to the internal dissimilarity or similarity of the thought masses whose reproduction is sought.

The form of the developing thought mass (or the quality of the image which accompanies its formation) depends upon the relative form of the motion[2] of the matter in which it is shaped, so that a given form of motion of the matter, causes a like form of the thought mass shaped within it; and conversely, whatever the form of the thought mass, it presupposes a like form of motion of the matter in which it is shaped.

All thought masses simultaneously being formed (in our cerebro-spinal system) are connected in consequence of a physical (chemical-electrical) process between the sites where they are formed.

Each thought mass strives to reproduce a thought mass of like form. It therefore tries to recreate the form of motion of the matter in which it is formed.

The assumption of mind as a unified carrier for that which

is enduring—produced by individual acts of mental life (images)—is based upon the following:

1. On the close connection and mutual interpenetration of all images. In order to explain the linking of a particular new image with others, it is however, *not sufficient* to simply assume a unified carrier; rather the cause as to why the given image enters into just such particular connections, with just such particular strengths, must be sought in the images to which it binds itself. Once these causes are given, however, it then becomes superfluous to make the assumption of a unified carrier for all of the images. . . .

Let us now apply these laws of mental processes, to which the explanation of our own inner perception leads, to explain what we perceive to be purposefulness on earth, i.e., to an explanation of existence and historical development.

For the explanation of our mental life, it was necessary to assume that the thought masses which were produced in our nervous system endure as part of our mind; that their internal relations persist without alteration, and that they are subjected to alteration only in so far as they enter into connection with other thought masses.

It is a direct consequence of these principles of explanation, that the minds of organic beings—i.e., the compact thought masses arising during their lives—also continue to exist after their death. (Their isolated continuance is not sufficient.) In order to explain the systematic development of organic nature, however—in which previously gathered experiences obviously serve as the foundation for subsequent creations—we must assume that these thought masses enter into a greater compact thought mass, the biosphere,[3] and there serve a higher mental life, according to the same laws as those which operate when we reproduce thought masses in our nervous system to serve our own mental life.

Take as an example, the case in which we see a red surface. The thought masses produced in an aggregate of individual primitive fibers is bound into a single, compact, thought mass, which enters into our thinking at once. In the same way, the thought masses produced in various individuals of a species of plant, which enter the biosphere from a region of the earth's surface which is not very diverse climatically, will be combined into a single impression. Just as various sense perceptions of the same object are united in our mind into one image of the object, so all plants of one part of the earth's surface will give the biosphere a picture, worked out in the finest detail, of its climatic and chemical condition. In this manner, the way in which the plan for later creations evolved from the earlier life of the earth, can be explained.

But, according to our principles of explanation, the continued existence of thought masses once present, requires no material carrier; yet all of the interconnections, at least every connection between thought masses of different kinds, can only occur by means of of the production of newer thought masses by a common process of the nervous system.

For reasons to be developed later, we can seek the carrier for a mental activity only in ponderable matter.

Now it is a fact, that the rigid crust of the earth, along with everything ponderable above it, does not serve a common "mental" process; we can only explain the movement of these ponderable substances by other causes.

Accordingly, the only remaining assumption is that the ponderable masses within the rigid crust of the earth are the carrier for the mental life of the earth.

Are these masses suitable for this purpose? What are the external conditions necessary for the life process? We can establish the foundation for an answer only empirically, on the basis of the living processes that are accessible to our observation; but only insofar as we succeed in explaining them, can we draw conclusions from them which are also applicable to other classes of phenomena.

Empirically, the external conditions of living processes in the range of phenomena accessible to us are:

Herbart on the Thought Process

Johann Friedrich Herbart (1776-1841)

Johann Friedrich Herbart, German philosopher and educational theorist, was the dominant influence on American education in the 1890s, until his classical theory was attacked by radical empiricist John Dewey in 1896.

The following passage from his seminal work, *Outlines of Educational Doctrine* (translated by Alexis F. Lange, New York: Macmillan, 1911) is typical of those upon which Riemann drew in formulating his theory of the process of creative discovery in terms of an elaboration of successively higher-dimensional, multiply connected manifolds.

Herbart writes (page 19):

> Each body of ideas is made up of complications of ideas, which, if the union is perfect, come and go in consciousness as undivided wholes, and of series, together with their interlacings, whose members unfold successively, one by one, provided they are not checked. The closer the union of parts within these complications and series, the more absolute the laws according to which ideas act in consciousness, the stronger is the resistance against everything opposing their movement; hence the difficulty of acting upon them through instruction. They admit, however, of additions and recombinations, and so may in the course of time undergo essential changes; up to a certain point they even change of themselves if repeatedly called into consciousness by dissimilar occasions, e.g., by the frequent delivery of the same lecture before different audiences.

—David Cherry

1. The higher and more completely developed the life-process, the more it is necessary to protect its carrier from external causes of motion which strive to change the relative position of its parts.

2. The physical processes (changes in matter) known to us that serve as a means for the thought process:

(a) absorption of gas by liquids

(b) osmosis inward through a cell wall

(c) formation and decomposition of chemical compounds

(d) Galvanic currents.

3. The substance of organisms has no recognizable crystalline structure; it is partly solid (only slightly brittle), partly gelatinous, partly liquid or gaseous, but always porous, that is, markedly penetrable by gases.

4. Among all chemical elements, only the four so-called organic elements are general carriers for the life process, and again, quite definite compounds of these, the so-called organizing compounds, are components of organic bodies (protein, cellulose, etc.).

5. Organic compounds exist only to a definite upper temperature limit, and can be carriers of life only to a definite lower one.

ad. 1. Changes in the relative position of the parts of a body are caused by the following (in decreasing stepwise order of their effect): mechanical forces, changes in temperature, light radiation; accordingly, we can order the facts—of which our proposition is the general expression—as follows:

1. The propagation of lower organisms through division. The gradually decreasing reproductive capacity of higher animal organisms.

2. The parts of plants are the more sensitive to changes in temperature, the more intensive and the more highly developed the life process is in them. In the higher animal organisms, an almost constant temperature governs, especially in their most vital parts.

3. The parts of the nervous system which serve independent thinking are protected against all these influences as much as possible.

Obviously, the foundation for the fact first presented[4] is that, the more the relative position of the parts can be determined by processes occurring within the interior of the matter, the less will it be determined by external motion. This independence from external sources of motion, however, occurs to a far higher degree inside the crust of the earth, than for organic beings on the outside.

In the context of the following facts, taken together, those placed under 4. and 5. [above] are apparently contrary to our assumption; they would be so, in fact, if absolute validity were to be ascribed to those conditions perceived by us for the possibility of a life process, rather than a merely relative validity within the limits of our experience. The following reasons go against their absolute validity, however:

1. All of nature, with the exception of the surface of the earth, would then have to be considered dead, since on all other celestial bodies, temperature and pressure relations predominate under which organic compounds cannot exist.

2. It is absurd to assume that the organic arose from the inorganic on the rigid crust of the earth. In order to explain the origin of the lowest organisms on the earth's crust, some organizing principle must be assumed, and thus a thought process[5] must exist under conditions in which organic compounds could not exist.

We must therefore assume that these conditions are valid only for the life process under the present relationships on the surface of the earth, and only in so far as we are successful in explaining these, can we judge from them the possibility of the life process governed by different relationships.

Why, therefore, are only the four organic elements universal carriers of the life process? The reason can only be sought in properties by which these four elements are distinguished from all others.

1. One such general property of these four elements consists in the fact that they and their compounds are the most difficult to condense of all materials, and, some of them have not yet been condensed at all.

2. Another property which they share is the great multiplicity of their compounds and the ease with which they decompose. This property, however, could just as well be the consequence of their use in living processes as its cause.

However, the former property, that of being difficult to condense, is what makes these four elements preeminently suited to serve life processes. To a certain extent this is directly explainable from the conditions of the life process enumerated under 2. and 3.,[6] but even more if we attempt to trace the phenomena found in the condensation of gases to liquids and solids, back to their causes. . . .

Zend-Avesta is in fact a life-giving word,* which creates new life for our mind, in knowledge as in faith. For like many a thought, which indeed was at one time powerfully effective in the course of development of mankind, but is now only preserved for us through tradition, *Zend-Avesta* arises now, all at once, from its apparent death, into a purer form of new life, and reveals new life in nature. Now as the life of nature—previously only manifest on the surface of the earth—is immeasurably extended before our eyes, it appears inexpressibly more sublime. What we considered as the seat of forces working senselessly and unconsciously, now appears as the workplace of the highest spiritual activity. What our great poet has portrayed with prescient inspiration as the goal, which hovered before the mind of the investigator, is now fulfilled in a wondrous way.

Just as Fechner in his *Nanna* seeks to demonstrate that plants possess the characteristics of mind,[7] so the point of departure for his reflections in *Zend-Avesta* is the teaching that stars share characteristics of mind. His method is not to abstract general laws through induction in order to apply and confirm these in the explanation of nature, but rather to reason by analogy. He compares the earth to our own organism, which we know has a mind. He does not merely one-sidedly investigate the similarities, but also does as much justice to the dissimilarities. In this way he obtains the result that all the similarities indicate that the earth is a being possessing characteristics of a mind, and that all of the dissimilarities indicate that it is a being with a mind of a far higher order than our own. The persuasive power of this presentation lies in its many-sided, detailed exposition. The total impression of the picture unfurled for us, of the life of the earth, provides evidence for his view, and compensates for that which the individual conclusions lack in rigor.

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* Compare Fechner, *Zend-Avesta*, Vol. 1, Preface, page V.

This evidence rests on the intuitive clarity of the image, and on its execution in the greatest possible detail. I would therefore believe myself to be doing harm to Fechner's view, were I to attempt to present here, in outline, the course he takes in his works. In the following discussion of Fechner's views, I will ignore the form in which they are presented, and consider only the substance, and thus take as a basis the former method, the abstraction of general laws by induction and their confirmation in the explanation of nature.

Let us ask first: From what do we conclude that something has a mind (the occurrence within it of a continuing, unified thinking process)? We are directly aware of our own mind, and with others (human beings and animals), we infer it from individual purposeful movements.

In general, wherever we trace a well-ordered purposefulness back to a cause, we seek this cause in a process of thought; we do not have another explanation. Thinking itself, however, I can only consider as a process which occurs within the interior of ponderable matter. As is evident to anyone who tries to analyze inner perception impartially, it is impossible to explain thinking on the basis of the motion of matter in space; however, the abstract possibility of such an explanation may be conceded here.

No one will deny that purposefulness is perceived on the earth. And so the question arises: Where are we to locate the thought process that is the cause of this purposefulness?

The concern here is only with conditioned purposes (those which take place within limited time and space); unconditional purposes find their explanation in an eternal Will (not produced in a process of thought). The only purposefulness whose cause we perceive is that of our own actions. It originates in willing the end and reflecting upon means.

If we find a body consisting of ponderable matter in which a lattice of continuing, related purposes and actions are completely realized, we can explain this purposefulness by means of a continuing, unified thought process, and this hypothesis will be the most probable if (1) the purposefulness is not completed merely in parts of the body and (2) no reason is present to seek the cause of that purposefulness in a larger whole of which the body is a part.

If we apply this to the purposefulness which we perceive in human beings, animals, and plants, then it follows that a part of this purposefulness is to be explained by a thought process which occurs within these bodies; another part, however, the purposefulness of the organism itself, is to be explained by a process of thinking in a larger whole.

The reasons for this are:

1. The purposefulness of organisms does not find completion in individual organisms. The reasons for the constitution of the human organism are obviously to be sought in the constitution of the entire surface of the earth, with organic nature taken into account.

2. The organism's activities repeat themselves innumerable times, in part simultaneously in different individuals, partly successively in the life of an individual or a generation. For the purposefulness which lies in them already *per se*, we need not assume a special cause in each case, but rather a common cause.

3. In the case of human beings and animals, their constitutions undergo no further development within the lifetime of the single individual, nor (in the case of plants and embryos) within the life of a single generation. Therefore, the cause of their *purposefulness* is not to be sought in a simultaneously continuing process of thought.

Apart from these aspects of (organic) purposefulness, there is still in man and animals, by common consent—and in plants in Fechner's view—a closed lattice of interpenetrating and variable relations of purpose and action; and this purposefulness is explained by the existence of a unified "thought process" within them.

These conclusions which we draw from our principles are confirmed through our inner perception.

According to the same principles, however, we must look for the reason behind the purposefulness which we perceive in organisms in a unified thought process occurring in the earth, on the following grounds:

(a) The relationships of purpose and action characteristic of organic life on earth cannot be separated into separate systems; on the contrary, everything is interlocked. They cannot therefore be explained as several particular thought processes, in various parts of the earth.

(b) There is no basis, as far as our experience goes, for seeking the reason for this purposefulness in a greater whole. All organisms are determined only for life on the earth. The condition of the earth's crust contains, therefore, all the (external) reasons needed to explain how they are organized.

(c) Organisms found on earth are individual. According to everything that experience teaches, we must assume that they are not replicated on other celestial bodies.

(d) They do not persist throughout the life of the earth. Instead, new, more perfect organisms are always appearing. We must therefore seek the cause in a thought process that is simultaneously ascending to higher levels.

The assumption of a biosphere is therefore a hypothesis for explaining the existence and the historical development of the organic world, from the standpoint of exact natural science, of a natural explanation from causes.

"When the body of the lower soul dies," Fechner says, "the higher soul takes it up from its perceptual life into its life of memory." The souls of deceased creatures are thus said to form the elements for the soul-life of the earth.

The various processes of thought seem to be principally distinguished by their temporal rhythm. If plants possess minds, so must hours and days be for them, what seconds are for us; the corresponding period of time for the earth mind encompasses many millennia, at least, for its outward activity. As far as the historical memory of mankind reaches, all movements of the inorganic crust of the earth are probably to be explained by mechanical laws.

Antinomies

Thesis	Antithesis
The finite, the representable.	Infinite, conceptual systems which lie at the boundary of the representable.
I.	
Finite time and space elements.	The continuous.

II.

Freedom, i.e., not the capacity to begin absolutely, but rather to decide between two or more given possibilities.	Determinism.
So that decision through choice be possible, despite the existence of fully determinate laws of the working of images, one must assume that the psychic mechanism itself has, or at least takes on, in its development, the characteristic of leading to the necessity of decision through choice.	No one, when acting, can give up the conviction that the future is partly determined by his action.

III.

A God who operates in time (governance of the universe).	A timeless, personal, omniscient, omnipotent, all-good God (providence).

IV.

Immortality.	A thing in itself, which is the basis of our transient existence, endowed with transcendental freedom, radical evil, intelligible character.

Freedom is entirely compatible with the strict lawfulness of the course of nature. But the concept of a timeless God is not tenable beside it. Rather, the limitation which omnipotence and omniscience must suffer through the freedom of creatures, in the sense established above, is removed through the assumption of a God operating in time, who is a guide for the hearts and fate of man; the concept of providence must be supplemented and in part replaced by the concept of the governance of the universe.

General Relationship between the Conceptual Systems of Thesis and Antithesis

The method, which Newton used for founding the infinitesimal calculus, and which, since the beginning of this century, has been acknowledged by the best mathematicians as the only one which produces reliable results, is the method of limits. The method consists in this, that instead of considering a continuous transition from one value of a magnitude to another, from one position to another, or in general, from one mode of determination of a concept to another, one first considers a transition through a finite number of intermediate steps, and then allows the number of these intermediate steps to grow, so that the distance between two consecutive intermediate steps decreases *ad infinitum*.

Conceptual systems of antithesis are concepts indeed firmly determined through negative predicates, but not positively representable.

Just because an exact and complete representation of these conceptual systems is impossible, they are not accessible to direct investigation and treatment by our reflection. But they can be considered to lie at the boundary of the representable, i.e., one can form a conceptual system which lies within the representable, but which passes over into the given conceptual system through mere changes in the relative magnitudes. Apart from the relative magnitudes, the conceptual system remains unchanged in the transition to the limit. In the limiting case itself, however, some of the correlative concepts of the system lose their representability, in fact precisely those which mediate the relationship with other concepts.

II. Epistemological Issues

Attempt at a Theory of the Fundamental Concepts of Mathematics and Physics as the Foundation for the Explanation of Nature

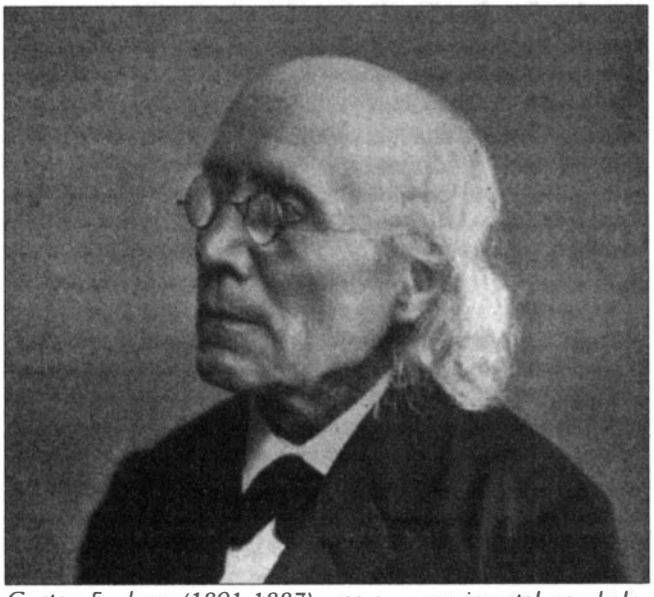
Natural science is the attempt to understand nature by means of exact *concepts*.

According to the concepts through which we comprehend nature, our perceptions are supplemented and filled in, not simply at each moment, but also future perceptions are seen as necessary. Or, to the degree that the conceptual system is not fully sufficient, future perceptions are determined beforehand as probable; according to the concepts, what is "possible" is determined (thus also what is "necessary," and conversely, impossible). And the degree of possibility (of "probability") of each individual event which is seen as possible, in light of these concepts, can be mathematically determined, if the concepts are precise enough.

To the extent that what is necessary or probable, according to these concepts, takes place, then this confirms the concepts, and the trust that we place in these concepts rests on this confirmation through experience. But if something takes place that is unexpected according our existing assumptions, i.e., that is impossible or improbable according to them, then the task arises of completing them or, if necessary, reworking the axioms, so that what is perceived ceases to be impossible or improbable. The completion or improvement of the conceptual system forms the "explanation" of the unexpected perception. Our comprehension of nature gradually becomes more and more complete and correct through this process, simultaneously penetrating more and more behind the surface of appearances.

The history of causal natural science, in so far as we can trace it back, shows that this is, in fact, the way our knowledge of nature advances. The conceptual systems that are now the basis for the natural sciences, arose through a gradual transformation of older conceptual systems, and the reasons that drove us to new modes of explanation can always be traced back to contradictions and improbabilities that emerged from the older modes of explanation.

The formation of new concepts, in so far as this process is



Gustav Fechner (1801-1887) was an experimental psychologist and professor of physics at the University of Leipzig from 1834 until 1839, when he resigned because of illness. His work, however, continued to be very wide-ranging after his subsequent recovery. He is remembered today chiefly in connection with Fechner's (or Weber's) law that stimuli are perceived by the mind with logarithmic compression: The intensity of a sensation increases arithmetically if the intensity of the stimulus increases geometrically.

accessible to observation, therefore takes place in this way.

Herbart furnished the proof that concepts that allow us to comprehend the world—those whose origin we can trace neither in history nor in our own development, because they are delivered to us unnoticed through our language—can be derived from this source, in so far as they are more than mere forms combining simple sense images; and therefore these concepts need not be derived from some special constitution of the human mind which precedes all experience (such as Kant's categories).

This proof of their origin in our ability to comprehend that which is given to us by sense perception, is important for us, *because it is only in this way that their meaning can be determined in a manner satisfactory for science*. . . .

After the concept of things existing in themselves has been formed, then in reflecting on the process of change, which contradicts the concept of things existing in themselves, the task arises of maintaining this already proven concept as far as possible. From this problem arise simultaneously the concepts of continuous change and causality.

All that is observed is the transition of a thing from one state into another, or, to speak more generally, from one mode of determination to another, without a sudden jump being perceived in the transition. In order to complete the observations, we can either assume that the transition occurs through a very great, but finite, number of leaps imperceptible our senses, or that the thing goes continuously through *all* of the intermediate steps, taking it from one state to the other. The strongest reason for the latter conception is the demand to maintain as far as possible, the already proven concept of the existence of the thing in itself. Of course, it is not possible to actually represent such a transition through *all* intermediate steps, which, however, as noted, is valid, strictly speaking, for all concepts.

At the same time, however, according to the concept of the thing in itself, formed earlier and proven by experience, the thing would remain what it is, unless something else intervened. This creates the impulse to seek a cause for every change.

I. When is our comprehension of the world true?

"When the relations among our conceptions correspond to the relations of things."

The elements of our picture of the world are completely distinct from the corresponding elements of the reality which they picture. They are something within us; the elements of reality are something outside of ourselves. But the connections among the elements in the picture, and among the elements of reality which they depict, must agree, if the picture is to be true.

The truth of the picture is independent of its degree of fineness; it does not depend upon whether the elements of the picture represent larger or smaller aggregates of reality. But, the connections must correspond to one another; a direct action of two elements upon each other may not be assumed in the picture, where only an indirect one occurs in reality. Otherwise the picture would be false and would need correction. If, however, an element of the picture is replaced by a group of finer elements, so that its properties emerge, partly from the simpler properties of the finer elements, but partly from their connections, and thus become in part comprehensible, then this increases our insight into the connection of things, but without the earlier understanding having to be declared false.

II. How do we find the relations among things?

"From the connections of phenomena."

The representation in determinate space-and-time relations of things of the senses is something met with in deliberate reflection on nature or is *given* in that reflection. However, as we well know, the *quality* of the characteristics of things of the senses—color, sound, tone, smell, taste, heat or cold, is something merely derived from our own sensations and does not exist outside of ourselves.

The relations among things must therefore become known to us from *quantitative* relations, the spatial and temporal relations of things of the senses and the relative intensities of their characteristics and their qualitative differences.

Knowledge of the connections among things must arise from reflection on the observed relations of these relations of magnitudes.

Causality

I. What an action strives to accomplish must be determined through the concept of the action; its acting cannot be dependent upon anything else than the action's own being.

II. This demand is satisfied when the action strives to maintain or restore *itself*.

III. Such an action is not conceivable, however, if the action is a thing, a being; but only if it is a state or a relationship. If a

striving exists, to maintain or restore something, then deviations from this something must also be possible—and indeed in different degrees. And in so far as this striving conflicts with other strivings, it will in fact be maintained or restored only to the extent possible. But there is no gradation of being; a difference of degrees is conceivable only for states or relationships. If therefore, an action strives to maintain or restore *itself,* it must be a state or a relationship.

IV. Obviously, such action can only occur in those things that can assume such a state. But in which of these things it occurs, and whether it occurs in them at all cannot be determined from the concept of the action.*

Kant quite rightly notes that we can neither discover the existence of a thing, nor that it is the cause of something else, merely from analysis of the concept of the thing; so that the concepts of being and causality cannot be derived from analysis but only from *experience.* When, however, he later believes himself compelled to assume that the concept of causality precedes all experience, this is tantamount to throwing the baby out with the bath; because this implies that the mind would be preconditioned to accept any perception, given by experience, as a cause, if it could be connected to *any other arbitrary one* as effect, according to a rule of mere sequence. (Of course, we must derive the relationships of causality from experience, but we must not dispense with correcting and completing our comprehension of the data of experience through reflection.)

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* These theses are valid only if the effect is to be ascribed to a simple real cause.

If two things a and *b* are connected through an external cause, then a consequence *c* can be ascribed either to the connection, the process of being connected itself, or else to a change in the degree of the connection. The simplest assumption is that the consequence *c* can be ascribed to the process of being connected.

It is unnecessary to to take these considerations further. Their principle consists in holding to the thesis: "What an action strives to effect must be determined from the concept of the action"; but this thesis must be applied, not as Leibniz or Spinoza did, to beings with a manifold of determinations, but rather to real causes of the greatest possible simplicity.

In German, one tends to translate "actio" as well as "effectus" by "Wirkung [effect]." Since the word occurs in the latter sense more commonly, unclarity easily arises if it is used for "actio," as, for example, with the standard translation of "actio aequalis est reactioni [action and reaction are equal]," or "principium actionis minimae [principle of least action]." Kant seeks to remedy this by adding the Latin expressions "actio" and "actio mutua" in parenthesis to "Wirkung" and "Wechselwirkung [interaction]." One could perhaps write, "die Kraft is gleich der Gegenkraft [the force is equal to the opposing force]," "Satz vom kleinsten Kraftaufwande [the principle of least expenditure of force]." Since, in fact, we lack a simple expression for "agere," a striving directed toward something else, I may be permitted the use of the foreign word [agens, action].

The word hypothesis now has a somewhat different meaning than with Newton. We are now accustomed to understand by hypothesis all that is added by thought to phenomena.

Newton was far from the absurd thought that the explanation of phenomena could be gained by abstraction from experience.

Newton: [In Latin from the General Scholium of *Principia Mathematica*] "And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to natural philosophy. [. . .] But hitherto I have not been able to discover the cause of those properties of gravity from phenomena, and I frame no hypotheses."

Arago, *Oeuvres Complètes,* Vol. 3, 505:

[In French] "Once and once only did Laplace rise into the realm of conjecture. His conception at that time was nothing less than a cosmogony."

Laplace in response to Napoleon's question, why the name God did not occur in his *Celestial Mechanics:* [in French] "Sire, I have no need for that hypothesis."

The distinction that Newton makes between laws of motion, or axioms, and hypotheses, does not seem tenable to me. The law of inertia is the hypothesis: If a material point were present alone in the world and moved in space with a definite velocity, then it would constantly maintain this velocity.

III. Natural Philosophy

1. Molecular Mechanics

The free movement of a system of material points m_1, m_2 . . . with rectangular coordinates x_1, y_1, z_1; x_2, y_2, z_2; . . . , on which forces X_1, Y_1, Z_1; X_2, Y_2, Z_2; . . . act in parallel to the three axes, takes place according to the equations

$$(1) \qquad m_\iota \frac{d^2x_\iota}{dt^2} = X_\iota , \quad m_\iota \frac{d^2y_\iota}{dt^2} = Y_\iota , \quad m_\iota \frac{d^2z_\iota}{dt^2} = Z_\iota .$$

This law can also be expressed as follows: The accelerations are so determined that

$$\sum m_\iota \left(\left(\frac{d^2x_\iota}{dt^2} - \frac{X_\iota}{m_\iota} \right)^2 + \left(\frac{d^2y_\iota}{dt^2} - \frac{Y_\iota}{m_\iota} \right)^2 + \left(\frac{d^2z_\iota}{dt^2} - \frac{Z_\iota}{m_\iota} \right)^2 \right)$$

becomes a minimum; for this function of the accelerations takes its smallest value 0 if the accelerations collectively are determined in accordance with equation (1), that is, the magnitudes $\frac{d^2x_\iota}{dt^2} - \frac{X_\iota}{m_\iota} \cdots$ collectively = 0, and they also take the minimum value only then; for, were one of these magnitudes, for example, $\frac{d^2x_\iota}{dt^2} - \frac{X_\iota}{m_\iota}$ not equal to 0, then $\frac{d^2x_\iota}{dt^2}$ could continuously change so that the absolute value of this magnitude and consequently its square would decrease. The function would thus become smaller if all the other accelerations were simultaneously left unchanged.

This function of the accelerations is distinguished from

$$\sum m_\iota \left(\left(\frac{d^2x_\iota}{dt^2} \right)^2 + \left(\frac{d^2y_\iota}{dt^2} \right)^2 + \left(\frac{d^2z_\iota}{dt^2} \right)^2 \right) - 2 \sum \left(X_\iota \frac{d^2x_\iota}{dt^2} + Y_\iota \frac{d^2y_\iota}{dt^2} + Z_\iota \frac{d^2z_\iota}{dt^2} \right)$$

only by a constant, that is, by a magnitude independent of the accelerations.

If the forces between points result only from attraction and repulsion, which are functions of distance, and the ιth point and the ι'th point at a distance r repulse one another with a force $f_{\iota,\iota'}(r)$ or attract one another with the force $-f_{\iota,\iota'}(r)$ then, as is known, the components of the forces can be expressed through the partial derivatives of a function of the coordinates of all the points

$$P = \sum_{\iota,\iota'} F_{\iota,\iota'}(r_{\iota,\iota'}),$$

where $F_{\iota,\iota'}(r)$ is a function with derivative $f_{\iota,\iota'}(r)$, and for ι and ι' two different indices are set for each.

If these values of the components

$$X_\iota = \frac{\partial P}{\partial x_\iota}, \quad Y_\iota = \frac{\partial P}{\partial y_\iota}, \quad Z_\iota = \frac{\partial P}{\partial z_\iota}$$

are substituted into the above function of the accelerations and are multiplied by $\frac{dt^2}{4}$, through which the positions of their maxima and minima are not changed, then we obtain an expression which is distinguished from

$$\frac{1}{4}\sum\left(\left(d\frac{dx_\iota}{dt}\right)^2 + \left(d\frac{dy_\iota}{dt}\right)^2 + \left(d\frac{dz_\iota}{dt}\right)^2\right) - P_{(t+dt)}$$

only by a magnitude which is independent of the accelerations. If the position and the velocities of the points at time t are given, then this position is determined at time $t + dt$ such that this magnitude becomes as small as possible. Accordingly, there is a striving for this magnitude to become a minimum.

This law can be explained on the basis of actions which strive to make the individual terms of this expression as small as possible if we assume *that the strivings working against one another are so equalized that the sum of the magnitudes which the individual actions strive to maintain at a minimum, becomes itself a minimum.*

If we assume that the masses of the points $m_1, m_2, \ldots, m_n$ behave like the whole numbers $k_1, k_2, \ldots, k_n$, so that $m_\iota = k_\iota\mu$, then the expression, which becomes as small as possible, consists of the sum of the magnitudes

$$\frac{\mu}{4}\left(\left(d\frac{d^2x_\iota}{dt}\right)^2 + \left(d\frac{dy_\iota}{dt}\right)^2 + \left(d\frac{dz_\iota}{dt}\right)^2\right)$$

for the totality of material particles μ and of magnitude $-P_{t+dt}$. If we therefore, with Gauss, consider the magnitude

$$\left(d\frac{d^2x_\iota}{dt}\right)^2 + \left(d\frac{dy_\iota}{dt}\right)^2 + \left(d\frac{dz_\iota}{dt}\right)^2$$

as the measure of the deviation of the state of motion of mass μ at time $t + dt$ from its state of motion at time t, then the analysis of the total action in relation to each mass yields an action which strives to make the deviation of its state of motion at time $t + dt$ as small as possible relative to its state of motion at time t, or an effort to preserve its state of motion, and, additionally, an action which strives to keep the magnitude $-P$ as small as possible.

The latter action can be analyzed into efforts to keep the individual terms of the sum $\sum_{\iota,\iota'} F_{\iota,\iota'}(r_{\iota,\iota'})$ as small as possible, that is, into attractions and repulsions between any two points, and this would lead us back to the customary explanation of the laws of motion from the law of inertia and of attraction and repulsion; but it can also lead us back, for all natural forces known to us, to the forces that act between contiguous spatial elements, as will be explained in the following article on gravitation.

2. New Mathematical Principles of Natural Philosophy*

Although the title of this essay will hardly create a favorable impression on most readers, it nonetheless seems to me to best express the overall direction of the essay. Its purpose is to penetrate beyond the foundations of astronomy and physics laid by Galilei and Newton, into the interior of nature. For astronomy, certainly these speculations cannot immediately have any practical use, but I hope that this circumstance will not cause any diminution of interest in the eyes of the readers of this publication. . . .

The foundation for those general laws of the motion of ponderable bodies that are presented at the beginning of Newton's *Principia* lies in the internal state of these bodies. Let us attempt to form an analogy between these and our own inner mode of perception. New image masses constantly arise in us and very rapidly disappear again from our consciousness. We observe a constant activity of our mind. Every mental act is based upon something enduring, which is manifest (through memory) on certain occasions, without exerting a lasting influence on the phenomena. Thus (with every act of thinking) something enduring continually enters our mind, which does not however, exert a lasting influence upon the world of phenomena. Every mental act, therefore, is based upon something enduring, which enters our mind with the act, but at the same moment completely disappears from the world of phenomena.

Guided by this fact, I form the hypothesis that there is a kind of space-filling substance which continually flows into ponderable atoms and there disappears from the world of phenomena (the corporeal world).[8]

Both hypotheses can be replaced by the one, that in all ponderable atoms, substance from the corporeal world continuously enters into the world of mind. The reason the substance disappears there is to be sought in the thought matter which was formed in the immediately preceding period; and the ponderable bodies are accordingly the place where the world of mind engages the corporeal world.†

The effect of universal gravitation, the first thing to be explained by this hypothesis, is well known to be fully determined for every part of space, if the potential function P of all ponderable mass for this part of space be given, or, which is the same

* Discovered on March 1, 1853.

† At every instant, a definite quantity of substance, proportional to the gravitational force, enters into every ponderable atom, and disappears there.

It is a consequence of the psychology based on Herbart's work, that substantiality accrues not to the mind but to every individual image formed within it.

thing, there is a function of position P, such that the ponderable masses contained within the closed surface S, are $\frac{1}{4\pi}\int \frac{\partial P}{\partial p}\, dS$.

If we now assume that the substance that fills space is an incompressible homogeneous fluid, without inertia, and that an amount proportional to the mass of any given atom flows into it during equal times, then obviously, the pressure exerted on the ponderable atom (will be proportional to the velocity of the substance at the site of the atom(?))[9]

Thus the effect of universal gravitation on a ponderable atom can be expressed through (and thought of as dependent upon) the pressure of this space-filling substance in the immediate neighborhood of the atom.

It necessarily follows from our hypothesis that the space-filling substance must propagate the vibrations that we perceive as light and heat.

If we consider a simple polarized beam, and designate as x the distance of an indeterminate point of this beam from a fixed origin, and y its displacement at a time t, then the following equation must be at least very nearly satisfied, since the velocity of propagation of the vibrations in space free of ponderable atoms is under all conditions very nearly constant (= α):

$$y = f(x + \alpha t) + \varphi(x - \alpha t).$$

For it to be strictly satisfied,

$$\frac{\partial y}{\partial t} = \alpha\alpha \int^{t} \frac{\partial^2 y}{\partial x^2}\, d\tau$$

would have to apply; obviously, however, for the sake of experiment, we can be satisfied with the equation

$$\frac{\partial y}{\partial t} = \alpha\alpha \int^{t} \frac{\partial^2 y}{\partial x^2}\, \varphi\,(t - \tau)\, d\tau$$

even if $\varphi(t - \tau)$ is not equal to 1 for all positive values of $t - \tau$ (which decreases *ad infinitum* with increasing $t - \tau$), as long as for a sufficiently long period of time it remains very close to 1. . . .

Let the positions of the points of the substance at a given time t be expressed by a rectilinear coordinate system and let the coordinates of an indeterminate point O be x, y, z. Similarly, let the coordinates of a point O' be x', y', z', also with regard to a rectilinear coordinate system. Then x', y', z' are functions of x, y, z, and $ds'^2 = dx'^2 + dy'^2 + dz'^2$ will be equal to a homogeneous quadratic expression of dx, dy, dz. According to a well-known theorem, the linear expressions of dx, dy, dz

$$\begin{aligned}
\alpha_1 dx + \beta_1 dy + \gamma_1 dz &= ds_1 \\
\alpha_2 dx + \beta_2 dy + \gamma_2 dz &= ds_2 \\
\alpha_3 dx + \beta_3 dy + \gamma_3 dz &= ds_3
\end{aligned}$$

can now always in one and only one way be determined, such that

$$dx'^2 + dy'^2 + dz'^2 = G_1^2 ds_1^2 + G_2^2 ds_2^2 + G_3^2 ds_3^2$$

while

$$ds^2 = dx^2 + dy^2 + dz^2 = ds_1^2 + ds_2^2 + ds_3^2\,.$$

The magnitudes $G_1 - 1$, $G_2 - 1$, $G_3 - 1$ then signify the major deformations for the particle of substance at O, in the transition from the former form to the latter. I indicate them by $\lambda_1, \lambda_2, \lambda_3$.

Now I assume that a force results from the difference between the earlier forms of the particle of substance and its form at time t, which strives to change it; and, other things being equal, that the influence of an earlier form will become the less the longer the time prior to t when it occurred. Thus there is a limit before which all earlier forms can be ignored. I further assume that those states that still manifest a detectable influence differ so slightly from the state at time t, that the deformations may be regarded as infinitely small. The forces that strive to make $\lambda_1, \lambda_2, \lambda_3$ small can then be regarded as linear functions of $\lambda_1, \lambda_2, \lambda_3$; and indeed, because of the homogeneity of the aether for the total moment of these forces (the force which strives to make λ_1 small must be a function of $\lambda_1, \lambda_2, \lambda_3$, which remains unchanged when we exchange λ_2 with λ_3, and the remaining forces must follow from it, when λ_2 is exchanged with λ_1, and λ_3 with λ_1) we obtain the following expression:

$$\begin{aligned}
&\delta\lambda_1(a\lambda_1 + b\lambda_2 + b\lambda_3) + \delta\lambda_2(b\lambda_1 + a\lambda_2 + b\lambda_3) \\
&\quad + \delta\lambda_3(b\lambda_1 + b\lambda_2 + a\lambda_3)
\end{aligned}$$

or with a somewhat changed meaning of the constants:

$$\begin{aligned}
&\delta\lambda_1\big(a(\lambda_1 + \lambda_2 + \lambda_3) + b\lambda_1\big) + \delta\lambda_2\big(a(\lambda_1 + \lambda_2 + \lambda_3) + b\lambda_2\big) \\
&\qquad + \delta\lambda_3\big(a(\lambda_1 + \lambda_2 + \lambda_3) + b\lambda_3\big) \\
&= \tfrac{1}{2}\,\delta\big(a(\lambda_1 + \lambda_2 + \lambda_3)^2 + b(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\big).
\end{aligned}$$

Now the moment of the force that strives to change the form of the infinitely small particle of substance at O, can be regarded as resulting from forces that strive to change the length of the line elements ending at O. We therefore arrive at the following law of action: If dV is the volume of an infinitely small particle of substance at point O and time t, and dV' the volume of the same particle at time t', then the force resulting from the difference in the two states of the substance, which strives to elongate ds, is expressed by

$$a\,\frac{dV - dV'}{dV} + b\,\frac{ds - ds'}{ds}\,.$$

The first part of this expression derives from the force with which a particle of substance resists a change in volume without a change of form, the second from the force with which a physical line element resists a change in length.

Now there is no reason to assume that the effects of both causes change with time in accordance with the same law; thus if we sum the effects of all earlier forms of a particle of substance upon the change of the line element ds at time t, then the value of $\frac{\delta ds}{dt}$, which they strive to determine, becomes

$$= \int_{-\infty}^{t} \frac{dV' - dV}{dV}\, \psi\,(t - t')\, \delta t' + \int_{-\infty}^{t} \frac{ds' - ds}{ds}\, \varphi\,(t - t')\, \delta t'\,.$$

How then must the functions ψ and φ be constituted such that

gravitation, light, and radiant heat may be propagation through the substance of space?

The effects of ponderable matter upon ponderable matter are:

(1) Attractive and repulsive forces inversely proportional to the square of the distance.

(2) Light and radiant heat.

Both classes of phenomena can be explained if we assume that the entirety of infinite space be filled with a homogeneous substance and that every particle of that substance acts directly only upon its immediate neighborhood.

The mathematical law in accordance with which this occurs can be thought of as divided into

(1) the resistance of a particle of substance to a change in volume, and

(2) the resistance of a physical line element to a change in length.

Upon the first part are founded gravitation and electrostatic attraction and repulsion; upon the second, the propagation of light and heat, and electrodynamic or magnetic attraction and repulsion.

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3. Gravitation and Light

The Newtonian explanation of gravitational motion and the motions of celestial bodies consists in the assumption of the following causes:

1. There exists an infinite space with the properties which are assigned to it by geometry, and there exist ponderable bodies which change their positions within this space only continuously.

2. At every mass-point, there is at every moment a cause determined by magnitude and direction, by virtue of which cause the mass-point has a determinate motion (matter in a determinate state of motion). The measure of this cause is velocity.*

The phenomena to be explained here do not yet lead to the assumption of different masses for ponderable bodies.

3. At every point of space, there exists at every moment a cause (accelerating force), determined by magnitude and direction, which communicates a determinate motion to every mass point present, and indeed, the same motion to each, which combines geometrically with the motion that it already has.

4. At every mass-point in space, there exists a cause (absolute gravity) determined by magnitude, which combines geometrically with all other accelerating forces present there. By virtue of this cause, at every point of space an accelerating force exists, inversely proportional to the square of its distance from this mass-point and directly proportional to its gravitational force.†

The cause, determined according to magnitude and direction (accelerating gravitational force), which, according to 3., is found at every point in space, I seek in the form of motion of a substance that is continuously spread through all infinite space, and, indeed, I assume that the direction of the motion is equal to the direction of the force from which it is to be explained, and the velocity is proportional to the magnitude of the force. This substance can therefore be represented as a physical space whose points move in geometrical space.

According to this assumption, all effects caused by ponderable bodies on ponderable bodies through empty space must be propagated by this substance. Therefore also the forms of motion of which light and heat consist, which celestial bodies transmit to one another, must be forms of motion of this substance. These two phenomena, however, gravitation and the motion of light through empty space, are the only ones that must be explained *purely* by means of the motions of this substance.

Now I assume that the actual motion of the substance in empty space is combined from the motion which must be assumed for explanation of gravitation and that which must be assumed for the explanation of light.

The further development of this hypothesis can be divided into two parts in that the following are to be sought:

1. The laws of motion of the substance which must be assumed for the explanation of the phenomena.

2. The causes by means of which these motions can be explained.

The first subject is mathematical, the second, metaphysical. In reference to the latter, I note in advance that the goal will not be considered to be any explanation on the basis of causes that strive to change the distance between two points of the substance. This method of explanation by means of attractive and repulsive forces owes its general application in physics not to any direct evidence (or specific conformity to reason), nor, apart from electricity and gravity, to its particular facility, but on the contrary, to the circumstance that the Newtonian law of attraction, in contradiction to the opinion of its discoverer, has so far been considered to need no further explanation.§

I. Laws of motion of the substance that, according to our assumption, causes the phenomena of gravitation and light.

Expressing the position of a point in space by means of rectilinear coordinates x_1, x_2, x_3, I designate the velocity components—parallel to the coordinates at time t—of the motion that causes the gravitational phenomena as u_1, u_2, u_3, and those of the motion that causes the phenomena of light as w_1, w_2, w_3, and those of the actual motion as v_1, v_2, v_3, so that $v = u + w$. As will emerge from the laws of motion themselves, the substance, if it is everywhere equally dense at one point in time, maintains this same density everywhere at all times. I will therefore assume this to be everywhere equal to 1 at time t.

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* Every material body, if alone in space, would either not change its position in space or would move in a straight line with constant velocity.

This law of motion cannot be explained by means of the Principle of Sufficient Reason: That the body continues its motion, must have a cause, which can only be sought in the internal state of the matter.

† The same mass point would undergo changes in motion between two points, whose directions coincide with the directions of the forces and whose magnitudes are proportional to the forces.

The force divided by the change in motion, therefore, always gives the same quotient for the same mass-point. This quotient is different for different mass-points and is called their mass.

§ [In English] Newton says: "That gravity should be innate, inherent, and essential to matter, so that one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man who has in philosophical matters a competent faculty of thinking can ever fall into it." See the third letter to Bentley.

a. Motion That Causes Only Gravitational Phenomena.

The gravitational force is determined at every point by the potential function V, whose partial derivatives $\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{x_3 \partial}$ are the components of the gravitational force, and this V is in turn determined through the following conditions (disregarding an additional constant):

1. $dx_1\, dx_2\, dx_3 \left(\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2}\right)$ outside the attracting body = 0, and has for every ponderable material element a constant value. This is the product of -4π in the absolute magnitude of the attractive force, which according to the theory of attraction must be assigned to it, and will be designated as dm.

2. If all attracting bodies are within a finite space, $r\frac{\partial V}{\partial x_1}, r\frac{\partial V}{\partial x_2}, r\frac{\partial V}{\partial x_3}$ at an infinite distance r from a point in this space are infinitely small.

Now according to our hypothesis, $\frac{\partial V}{\partial x} = u$ and consequently

$$dV = u_1 dx_1 + u_2 dx_2 + u_3 dx_3.$$

This includes the conditions

$$(1) \quad \frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2} = 0, \quad \frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3} = 0, \quad \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = 0,$$

$$(2) \quad \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right) dx_1\, dx_2\, dx_3 = -4\pi dm,$$

$$(3) \quad ru_1 = 0, \quad ru_2 = 0, \quad ru_3 = 0, \quad \text{for } r = \infty.$$

Conversely, the magnitudes u, if they satisfy these conditions, are equal to the components of the gravitational force. Since the conditions (1) contain the possibility of a function U from which arises the differential $dU = u_1 dx_1 + u_2 dx_2 + u_3 dx_3$ and thus the derivatives $\frac{\partial U}{\partial x} = u$, and the others then yield $U = V$ + constant.*

b. Motion that causes only light phenomena.

The motion that must be assumed in empty space for the explanation of the phenomena of light can be considered (following a theorem) as composed of plane waves, that is, of such motions where the form of motion is constant along each plane of a family of parallel planes (wave planes). Each of these wave systems consists then (in accord with observation) of motions parallel to the wave plane that are propagated perpendicular to the wave plane with a constant velocity c that is the same for all forms of motion (types of light).

If ξ_1, ξ_2, ξ_3 are the rectangular coordinates of a point in space for such a system of waves, the first being perpendicular, the others parallel to the wave plane, and $\omega_1, \omega_2, \omega_3$ are the components of velocity at this point parallel to the coordinates at time t, then we have

$$\frac{\partial \omega}{\partial \xi_2} = 0, \quad \frac{\partial \omega}{\partial \xi_3} = 0.$$

According to observation, first

$$\omega_1 = 0,$$

second, the movement is composed of motions with velocity c, one propagating from the positive side of the wave plane, and one propagating from the negative side. If the velocity components of the first are ω' and that of the latter are ω'', then the ω' remain unchanged if t increases by dt and ξ_1 increases by cdt, and the ω'' are unchanged, if t increases by dt and ξ_1 by $-cdt$, and we have $\omega = \omega' + \omega''$. From this it follows that

$$\left(\frac{\partial \omega'}{\partial t} + c\frac{\partial \omega'}{\partial \xi_1}\right) dt = 0, \quad \left(\frac{\partial \omega''}{\partial t} - c\frac{\partial \omega''}{\partial \xi_1}\right) dt = 0,$$

$$\frac{\partial^2 \omega'}{\partial t^2} = -c\frac{\partial^2 \omega'}{\partial \xi_1 \partial t} = cc\frac{\partial^2 \omega'}{\partial \xi_1^2}, \quad \frac{\partial^2 \omega''}{\partial t^2} = c\frac{\partial^2 \omega''}{\partial \xi_1 \partial t} = cc\frac{\partial^2 \omega''}{\partial \xi_1^2}$$

and thus

$$\frac{\partial^2 \omega}{\partial t^2} = cc\frac{\partial^2 \omega}{\partial \xi_1^2}.$$

These equations give the following symmetrical results:

$$\frac{\partial \omega_1}{\partial \xi_1} + \frac{\partial \omega_2}{\partial \xi_2} + \frac{\partial \omega_3}{\partial \xi_3} = 0,$$

$$\frac{\partial^2 \omega}{\partial t^2} = cc\left(\frac{\partial^2 \omega}{\partial \xi_1^2} + \frac{\partial^2 \omega}{\partial \xi_2^2} + \frac{\partial^2 \omega}{\partial \xi_3^2}\right),$$

which, expressed in the original coordinate system, become equations of the same form, that is,

$$(1) \quad \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} = 0,$$

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* This function U is therefore given through observation (from relative motions) by means of the general laws of motion, but only without taking account of a linear function of the coordinates, because we can only observe relative motions.

The determination of this function is based on the following mathematical theorem: A function V of position is determined within a finite space (ignoring a constant) if it is not said to be discontinuous along a surface, and for all of its elements $\left(\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2}\right) dx_1\, dx_2\, dx_3$ at the limit, either V or its derivative is given for an inward change of position, perpendicular to the limit. Of which it should be noted:

1. If this derivative at the bounding element ds is designated by $\frac{\partial V}{\partial p}$, then in the latter case $\int \sum \frac{\partial^2 V}{\partial x^2} dx_1\, dx_2\, dx_3$ must be equal to $-\int \frac{\partial V}{\partial p} ds$ through the entire space because of its bound; otherwise, in both cases, all of the determining elements can be taken arbitrarily and are therefore necessary to the determination.

2. For a spatial element where $\sum \frac{\partial^2 V}{\partial x^2}$ becomes infinitely large, the product of the two is to be substituted by $-\int \frac{\partial V}{\partial p} ds$ in relation to the limit of this element.

3. If $\sum \frac{\partial^2 V}{\partial x^2}$ has a value other than zero only within a finite space, then the boundary condition can be substituted by the statement that at an infinite distance R of a point in this space $R\frac{\partial V}{\partial x}$ becomes infinitely small.

(2) $$\frac{\partial^2 w}{\partial t^2} = cc\left(\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} + \frac{\partial^2 w}{\partial x_3^2}\right).$$

These equations are valid for every plane wave passing through the point (x_1, x_2, x_3) at time t and consequently also for the combined motion of all such plane waves.

c. Motion that causes both types of phenomena.

From the conditions established for u and w, the following conditions follow for v or laws of motion of the substance in empty space:

(I) $$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0,$$

$$\left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right)\left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}\right) = 0$$

(II) $$\left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right)\left(\frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_1}\right) = 0$$

$$\left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right)\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right) = 0,$$

as is easily derived if the operations are carried out.

These equations show that the motion of a point of the substance only depends on motions in contiguous regions of space and time, and their (complete) causes can be sought in the effects in their neighborhood.

Equation (I) proves our earlier assertion that the density of the substance remains unchanged during its motion; since

$$\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right) dx_1\, dx_2\, dx_3\, dt,$$

which as a result of this equation is equal to 0, expresses the mass of the substance which flows into the spatial element $dx_1\, dx_2\, dx_3$ in time element dt, and the mass of the substance contained in it therefore remains constant.

Conditions (II) are identical with the condition that

$$\left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right)(v_1 dx_1 + v_2 dx_2 + v_3 dx_3)$$

be equal to a complete differential dW. Now

$$\left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right)(w_1 dx_1 + w_2 dx_2 + w_3 dx_3) = 0$$

and consequently

$$dW = \left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right)(u_1 dx_1 + u_2 dx_2 + u_3 dx_3)$$

$$= \left(\partial^2 t - cc\left(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3\right)\right) dV$$

or, since $(\partial^2 x_1 + \partial^2 x_2 + \partial^2 x_3) dV = 0$,

$$= d\frac{\partial^2 V}{\partial t^2}.$$

. . . . . .

d. Common expression for the laws of motion of the substance and the effect of gravity on the motion of ponderable bodies.

The laws of these phenomena can be summed up by the condition that the variation of the integral

$$\frac{1}{2}\int\left[\sum\left(\frac{\partial \eta_1}{\partial t}\right)^2 - cc\left[\left(\frac{\partial \eta_2}{\partial x_3} - \frac{\partial \eta_3}{\partial x_2}\right)^2 + \left(\frac{\partial \eta_3}{\partial x_1} - \frac{\partial \eta_1}{\partial x_3}\right)^2 + \left(\frac{\partial \eta_1}{\partial x_2} - \frac{\partial \eta_2}{\partial x_1}\right)^2\right] dx_1\, dx_2\, dx_3\, dt\right]$$

$$+ \int V\left(\sum \frac{\partial^2 \eta_1}{\partial x_1 \partial t}\, dx_1\, dx_2\, dx_3 + 4\pi dm\right) dt$$

$$+ 2\pi \int dm \sum\left(\frac{\partial x_1}{\partial t}\right)^2 dt$$

becomes zero under appropriate boundary conditions.

In this expression, the first two integrals extend over the entire geometrical space, the latter over all elements of ponderable bodies, but the coordinates of every element of ponderable bodies are to be so determined as functions of time, and η_1, η_2, η_3, V as functions of x_1, x_2, x_3 and t, that a variation satisfying their boundary conditions produces only a variation of the second order of the integral.

Then the quantities $\frac{\partial \eta}{\partial t}$ $(=v)$ are equal to the velocity components of the motion of the substance and V is equal to the potential at time t at point (x_1, x_2, x_3).

Translator's Notes

1. The German expression is *Geistesmasse*. It had earlier appeared in the correspondence between Schiller and Goethe (personal communication of George Gregory).
2. The expression *form of motion* (*Bewegungsform*), which begins to appear here early in the fragments, appears as "forms of motion (types of light)" in one late occurrence in which the subject is electromagnetic radiation. This suggests that *form of motion* refers to wavelength or frequency.
3. In the fragments on psychology and metaphysics, Riemann refers to the *Erdseele*. The literal translation is *earth mind* or *earth soul*. We have instead used the expression *biosphere*. It will be helpful to the reader to keep in mind all the possibilities suggested by biosphere, earth mind, and earth soul, in the four instances where *biosphere* appears in the translation.

   The German *Seele* (soul or mind) is the equivalent of the Greek *psyche*. The Greek word also carries the meaning, *that which enables life*. In his *Harmonices Mundi*, Kepler used *anima*—the nearest Latin equivalent of *psyche*—as a metaphor for universal gravitation. The translator thanks George Gregory for these observations on the Greek and Latin terms and their use.
4. See the first of the three paragraphs marked "1" immediately preceding, which begins "1. The higher . . ."
5. The German word is *Denkprocess*.
6. Not the paragraphs 2. and 3. immediately preceding, but the earlier pair following the paragraph that reads, "Empirically, the external conditions of living processes in the range of phenomena accessible to us are:"
7. "Characteristics of mind" is used for *Beseeltheit*.
8. Here Riemann addresses the question of the space-filling substance, which he also calls "the aether" in one instance. In this translation, it is also referred to in the expression "particle of substance," and sometimes as simply "substance," after the concept of space-filling substance has been introduced. These expressions for space-filling substance are thus distinct from "ponderable atoms," "ponderable mass," or "ponderable bodies."
9. The question mark and both pairs of parentheses appear in the German without explanation. Are they Riemann's marks, or do they indicate an uncertain reading of the manuscript?