Chapter 4. The Curvature of Higher Dimensional Manifolds

A. An Inaugural Lecture

On June 10, 1854 the faculty of Göttingen University heard a lecture entitled Über die Hypothesen, welche der Geometrie zu Grunde liegen (On the Hypotheses which lie at the Foundations of Geometry). This lecture was delivered by Georg Friedrich Bernhard Riemann, who had been born just a year before Gauss' paper of 1827. Although the lecture was not published until 1866, the ideas contained within it proved to be the most influential in the entire history of differential geometry. To be sure, mathematicians had not neglected the study of surfaces in the meantime; in fact, Gauss' work had inspired a tremendous amount of work along these lines. But the results obtained in those years can all be proved with much greater ease after we have followed the long series of developments initiated by the turning point in differential geometry which Riemann's lecture provided.

A short account of the life and character of Riemann can be found in the biography by Dedekind* which is included in Riemann's collected works (published by Dover). His interest in many fields of mathematical physics, together with a demand for perfection in all he did, delayed until 1851 the submission of his doctoral dissertation Grundlagen für eine allegemeine Theorie der Functionen einer veränderlichen complexen Grösse (Foundations for a general theory of functions of a complex variable). Gauss' official report to the Philosophical Faculty of the University of Göttingen stated "The dissertation submitted by Herr Riemann offers convincing evidence of the author's thorough

*Even for those who can only plod through German, this is preferable to the account in E. T. Bell's Men of Mathematics, which is hardly more than a translation of Dedekind, written in a racy style and interladen with supercilious remarks of questionable taste.

and pentrating investigations in those parts of the subject treated in the dissertation, of a creative, active truly mathematical mind, and of a gloriously fertile originality."

Riemann was now qualified to seek the position of Privatdocent (a lecturer who received no salary, but was merely forwarded fees paid by those students who elected to attend his lectures). To attain this position he first had to submit an "inaugural paper" (Habilitationsschrift). Again there were delays, and it was not until the end of 1853 that Riemann submitted the Habilitationsschrift, Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe (On the representability of a function by a trigonometric series). Now Riemann still had to give a probationary inaugural lecture on a topic chosen by the faculty, from a list of three proposed by the candidate. The first two topics which Riemann submitted were ones on which he had already worked, and he had every reason to expect that one of these two would be picked; for the third topic he chose the foundations of geometry. Contrary to all traditions, Gauss passed over the first two, and picked instead the third, in which he had been interested for years. At this time Riemann was also investigating the connection between electricity, magnetism, light, and gravitation, in addition to acting as an assistant in a seminar on mathematical physics. The strain of carrying out another major investigation, aggravated perhaps by the hardships of poverty, brought on a temporary breakdown. However, Riemann soon recovered, disposed of some other work which had to be completed, and then finished his inaugural lecture in about seven more weeks.

Riemann hoped to make his lecture intelligible even to those members of the faculty who knew little mathematics. Consequently, hardly any formulas appear and the analytic investigations are completely suppressed. Although

Dedekind describes the lecture as a masterpiece of exposition, it is questionable how many of the faculty comprehended it. In making the following translation,* I was aided by the fact that I already had some idea what the mathematical results were supposed to be. The uninitiated reader will probably experience a great deal of difficulty merely understanding what Riemann is trying to say (the proofs of Riemann's assertions are spread out over the next several Chapters). We can be sure, however, that one member of the faculty appreciated Riemann's work. Dedekind tells us that Gauss sat at the lecture "which surpassed all his expectations, in the greatest astonishment, and on the way back from the faculty meeting he spoke to Wilhelm Weber, with the greatest appreciation, and with an excitement rare for him, about the depth of the ideas presented by Riemann."

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*The original is contained, of course, in Riemann's collected works. Two English translations are readily available, one in Volume 2 of Smith's _Source Book in Mathematics_ (Dover), and one in Clifford's _Mathematical Papers_ (Chelsea).

RIEMANN

ON THE HYPOTHESES WHICH LIE AT THE FOUNDATIONS OF GEOMETRY

Plan of the Investigation

As is well known, geometry presupposes the concept of space, as well as assuming the basic principles for constructions in space. It gives only nominal definitions of these things, while their essential specifications appear in the form of axioms. The relationship between these presuppositions [the concept of space, and the basic properties of space] is left in the dark; we do not see whether, or to what extent, any connection between them is necessary, or *a priori* whether any connection between them is even possible.

From Euclid to the most famous of the modern reformers of geometry, Legendre, this darkness has been dispelled neither by the mathematicians nor by the philosophers who have concerned themselves with it. This is undoubtedly because the general concept of multiply extended quantities, which includes spatial quantities, remains completely unexplored. I have therefore first set myself the task of constructing the concept of a multiply extended quantity from general notions of quantity. It will be shown that a multiply extended quantity is susceptible of various metric relations, so that Space constitutes only a special case of a triply extended quantity. From this however it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties which distinguish Space from other conceivable triply extended quantities can only be deduced from experience. Thus arises the problem of seeking out the simplest data from which the metric relations of Space can be determined, a problem which by its very nature is not completely determined, for there may be several systems of simple data which suffice to determine the metric relations of Space; for the present purposes, the most important system is that laid down as a foundation of geometry by Euclid. These data are — like all data — not logically necessary, but only of empirical certainty, they are hypotheses; one can therefore investigate their likelihood, which is certainly very great within the bounds of observation, and afterwards decide upon the legitimacy of extending them beyond the bounds of observation, both in the direction of the immeasurably large, and in the direction of the immeasurably small.

I. Concept of an n-fold extended quantity

In proceeding to attempt the solution of the first of these problems, the development of the concept of multiply extended quantity, I feel particularly entitled to request an indulgent criticism, as I am little practiced in these tasks of a philosophical nature where the difficulties lie more in the concepts than in the construction, and because I could not make use of any previous studies, except for some very brief hints on the subject which Privy Councillor Gauss has given in his second memoir on Biquadratic Residues, in the Göttingen Gelehrte Anzeige and in the Göttingen Jubilee-book, and some philosophical researches of Herbart.

1.

Notions of quantity are possible only when there already exists a general concept which admits particular instances. These instances form either a continuous or a discrete manifold, depending on whether or not a continuous transition of instances can be found between any two of them; individual instances are called points in the first case and elements of the manifold in the second. Concepts whose particular instances form a discrete manifold are so numerous that some concept can always be found, at least in the more highly developed languages, under which any given collection of things can be comprehended (and consequently, in the study of discrete quantities, mathematicians could unhesitatingly proceed from the principle that given objects are to be regarded as all of one kind). On the other hand, opportunities for creating concepts whose instances form a continuous manifold occur so seldom in everyday life that color and the position of sensible objects are perhaps the only simple concepts whose instances form a multiply extended manifold. More frequent opportunities for creating and developing these concepts first occur in higher mathematics.

Particular portions of a manifold, distinguished by a mark or by a boundary, are called quanta. Their quantitative comparison is effected in the case of discrete quantities by counting, in the case of continuous quantities by measurement. Measuring involves the superposition of the quantities to be compared; it therefore requires a means of transporting one quantity to be used as a standard for the others. Otherwise, one can compare two quantities only when one is a part of the other, and then only as to "more" or "less", not as to "how much". The investigations which can be carried out in this case form a general division of the science of quantity, independent of measurement, where quantities are regarded, not as existing independent of position and not as expressible in terms of a unit, but as regions in a manifold. Such investigations have become a necessity for several parts of mathematics, e.g., for the treatment of many-valued analytic functions, and the dearth of such studies is one of the principal reasons why the celebrated theorem of Abel and the contributions of Lagrange, Pfaff and Jacobi to the general theory of differential equations have remained unfruitful for so long. From this portion of the science of extended quantity, a portion which proceeds without any further assumptions, it suffices for the present purposes to emphasize two points, which will make clear the essential characteristic of an n-fold extension. The first of these concerns the generation of the concept of a multiply extended manifold, the second involves reducing position fixing in a given manifold to numerical determinations.

2.

In a concept whose instances form a continuous manifold, if one passes from one instance to another in a well-determined way, the instances through which one has passed form a simply extended manifold, whose essential characteristic is, that from any point in it a continuous movement is possible in only two directions, forwards and backwards. If one now imagines that this manifold passes to another, completely different one, and once again in a well-determined way, that is, so that every point passes to a well-determined point of the other, then the instances form, similarly, a double extended manifold. In a similar way, one obtains a triply extended manifold when one imagines that a doubly extended one passes in a well-determined way to a completely different one, and it is easy to see how one can continue this construction. If one consdiers the process as one in which the objects vary, instead of regarding the concept as fixed, then this construction can be characterized as a synthesis of a variability of $n+1$ dimensions from a variability of n dimensions and a variability of one dimension.

3.

I will now show, conversely, how one can break up a variability, whose boundary is given, into a variability of one dimension and a variability of lower dimension. One considers a piece of a manifold of one dimension — with a fixed origin, so that points of it may be compared with one another — varying so that for every point of the given manifold it has a definite value, continuously changing with this point. In other words, we take within the given manifold a continuous function of position, which, moreover, is not constant on any part of the manifold. Every system of points where the function has a constant value then forms a continuous manifold of fewer dimensions than the given one. These manifolds pass continuously from one to another as the function changes; one can therefore assume that they all emanate from one of them, and generally speaking this will occur in such a way that every point of the first passes to a definite point of any other; the exceptional cases, whose investigation is important, need not be considered here. In this way, the determination of position in the given manifold is reduced to a numerical determination and to the determination of position in a manifold of fewer dimensions. It is now easy to show that this manifold has $n-1$ dimensions, if the given manifold is an n-fold extension. By an n-time repetition of this process, the determination of position in an n-fold extended manifold is reduced to n numerical determinations, and therefore the determination of position in a given manifold is reduced, whenever this is possible, to a finite number of numerical determinations. There are, however, also manifolds in which the fixing of position requires not a finite number, but either an infinite sequence or a continuous manifold of numerical measurements. Such manifolds form, e.g., the possibilities for a function in a given region, the possible shapes of a solid figure, etc.

II. Metric relations of which a manifold of n dimensions is susceptible, on the assumption that lines have a length independent of their configuration, so that every line can be measured by every other

Now that the concept of an n-fold extended manifold has been constructed, and its essential characteristic has been found in the fact that position fixing in the manifold can be reduced to n numerical determinations, there follows, as the second of the problems proposed above, an investigation of the metric relations of which such a manifold is susceptible, and of the conditions which suffice to determine them. These metric relations can be investigated only in abstract terms, and their interdependence exhibited only through formulas. Under certain assumptions, however, one can resolve them into relations which are individually capable of geometric representation, and in this way it becomes possible to express the results of calculation geometrically. Thus, although an abstract investigation with formulas certainly cannot be avoided, the results can be presented in geometric garb. The foundations of both parts of the question are contained in the celebrated treatise of Privy Councillor Gauss on curved surfaces.

1.

Measurement requires an independence of quantity from position, which can occur in more than one way. The hypothesis which first presents itself, and which I shall develop here, is just this, that the length of lines is independent of their configuration, so that every line can be measured by every other.

If position-fixing is reduced to numerical determinations, so that the position of a point in the given n-fold extended manifold is expressed by n varying quantities x_1, x_2, x_3 and so forth up to x_n, then specifying a line amounts to giving the quantities x as functions of one variable. The problem then is, to set up a mathematical expression for the length of a line, for which purpose the quantities x must be thought of as expressible in units. I will treat this problem only under certain restrictions, and I first limit myself to lines in which the ratios of the quantities dx — the increments in the quantities x — vary continuously. One can then regard the lines as broken up into elements within which the ratios of the quantities dx may be considered to be constant. The problem then reduces to setting up a general expression for the line element ds at every point, an expression which will involve the quantities x and the quantities dx. I assume, secondly, that the length of the line element remains unchanged, up to first order, when all the points of this line element suffer the same infinitesimal displacement, whereby I simply mean that if all the quantities dx increase in the same ratio, the line element changes by the same ratio. Under these assumptions, the line element can be an arbitrary homogeneous function of the first degree in the quantities dx which remains the same when all the quantities dx change sign, and in which the arbitrary constants are functions of the quantities x. To find the simplest cases, I first seek an expression for the (n-1)-fold extended manifolds which are everywhere equidistant from the origin of the line element, i.e., I seek a continuous function of position which distinguishes them from one another. This must either decrease or increase in all directions from the origin; I will assume that it increases in all directions and therefore has a minimum at the origin. Then if its first and second differential quotients are finite, the first order differential must vanish and the second order differential cannot be negative; I assume that it is always positive. This differential expression of the second order remains constant if ds remains constant and increases quadratically when the quantities dx, and thus also ds, all increase in the same ratio; it is therefore equal to a constant times ds^2, and consequently ds equals the squareroot of an everywhere positive homogeneous function of the second degree in the quantities dx, in which the coefficients are continuous functions of the quantities x. In Space, if one expresses the location of a point by rectilinear coordinates, then $ds = \sqrt{\Sigma(dx)^2}$; Space is therefore included in this simplest case. The next simplest case would perhaps include the manifolds in which the line element can be expressed as the fourth root of a differential expression of the fourth degree. Investigation of this more general class would actually require no essentially different principles, but it would be rather time consuming and throw proportionally little new light on the study of Space, especially since the results cannot be expressed geometrically; I consequently restrict myself to those manifolds where the line element can be expressed by the square root of a differential expression of the second degree. One can transform such an expression into another similar one by substituting for the n independent variables, functions of n new independent variables. However, one cannot transform any expression into any other in this way; for the expression contains $n\frac{n+1}{2}$ coefficients which are arbitrary functions of the independent variables; by the introduction of new variables one can satisfy only n conditions, and can therefore make only n of the coefficients equal to given quantities. There remain $n\frac{n-1}{2}$ others, already completely determined by the nature of the manifold to be represented, and consequently $n\frac{n-1}{2}$ functions of position are required to determine its metric relations. Manifolds, like the Plane and Space, in which the line element can be brought into the form $\sqrt{\Sigma dx^2}$ thus constitute only a special case of the manifolds to be investigated here; they clearly deserve a special name, and consequently, these manifolds, in which the square of the lines element can be expressed as the sum of the squares of complete differentials, I propose to call flat. In order to survey the essential differences of the manifolds representable in the assumed form, it is necessary to eliminate the features depending on the mode of presentation, which is accomplished by choosing the variable quantities according to a definite principle.

2.

For this purpose, one constructs the system of shortest lines emanating from a given point; the position of an arbitrary point can then be determined by the initial direction of the shortest line in which it lies, and its distance, in this line, from the initial point. It can therefore be expressed by the ratios of the quantities dx^0, i.e., the quantities dx at the origin of this shortest line, and by the length s of this line. In place of the dx^0 one now introduces linear expressions $d\alpha$ formed from them in such a way that the initial value of the square of the line element will be equal to the sum of the squares of these expressions, so that the independent variables are: the quantity s and the ratio of the quantities $d\alpha$. Finally, in place of the $d\alpha$ choose quantities $x_1, x_2, \ldots, x_n$ proportional to them, but such that the sum of their squares equals s^2. If one introduces these quantities, then for infinitely small values of x the square of the line element equals Σdx^2, but the next order term in its expansion equals a homogeneous expression of the second degree in the $n\frac{n-1}{2}$ quantities $(x_1dx_2 - x_2dx_1)$, $(x_1dx_3 - x_3dx_1)$,..., and is consequently an infinitely small quantity of the fourth order, so that one obtains a finite quantity if one divides it by the square of the infinitely small triangle at whose vertices the variables have the values $(0,0,0,\ldots)$, $(x_1,x_2,x_3,\ldots)$, $(dx_1,dx_2,dx_3,\ldots)$. This quantity remains the same as long as the quantities x and dx are contained in the same binary linear forms, or as long as the two shortest lines from the initial point to x and from the initial point to dx remain in the same surface element, and therefore depends only on the position and direction of that element. It obviously equals zero if the manifold in question is flat, i.e., if the square of the line element is reducible to Σdx^2, and can therefore be regarded as the measure of deviation from flatness in this surface direction at this point. When multiplied by $-3/4$ it becomes equal to the quantity which Privy Councillor Gauss has called the curvature of a surface. Previously, $n\frac{n-1}{2}$ functions of position were found necessary in order to determine the metric relations of an n-fold extended manifold representable in the assumed form; hence if the curvature is given in $n\frac{n-1}{2}$ surface directions at every point, then the metric relations of the manifold may be determined, provided only that no identical relations can be found between these values, and indeed in general this does not occur. The metric relations of these manifolds, in which the line element can be represented as the square root of a differential expression of the second degree, can thus be expressed in a way completely independent of the choice of the varying quantities. A similar path to the same goal could also be taken in those manifolds in which the line element is expressed in a less simple way, e.g., by the fourth root of a differential expression of the fourth degree. The line element in this more general case would not be reducible to the square root of a quadratic sum of differential expressions, and therefore in the expression for the square of the line element the deviation from flatness would be an infinitely small quantity of the second dimension, whereas for the other manifolds it was an infinitely small quantity of the fourth dimension. This peculiarity of the latter manifolds therefore might well be called planeness in the smallest parts. For present purposes, however, the most important peculiarity of these manifolds, on whose account alone they have been examined here, is this, that the metric relations of the doubly extended ones can be represented geometrically by surfaces and those of the multiply extended ones can be reduced to those of the surfaces contained within them, which still requires a brief discussion.

3.

In the conception of surfaces, the inner metric relations, which involve only the lengths of paths within them, are always bound up with the way the surfaces are situated with respect to points outside them. We may, however, abstract from external relations by considering deformations which leave the lengths of lines within the surfaces unaltered, i.e., by considering arbitrary bendings — without stretching — of such surfaces, and by regarding all surfaces obtained from one another in this way as equivalent. Thus, for example, arbitrary cylindrical or conical surfaces count as equivalent to a plane, since they can be formed from a plane by mere bending, under which the inner metric relations remain the same; and all theorems about the plane — hence all of planimetry — retain their validity. On the other hand, they count as essentially different from the sphere, which cannot be transformed into the plane without stretching. According to the previous investigations, the inner metric relations at every point of a doubly extended quantity, if its line element can be expressed as the square root of a differential expression of the second degree, which is the case with surfaces, is characterized by the curvature. For surfaces, this quantity can be given a visual interpretation as the product of the two curvatures of the surface at this point, or by the fact that its product with an infinitely small triangle formed from shortest lines is, in proportion to the radius, half the excess of the sum of its angles over two right angles [that is, equal to the excess of the sum over π, when the angles are measured in radians]. The first definition would presuppose the theorem that the product of the two radii of curvatures is unaltered by mere bendings of a surface, the second, that at each point the excess over two right angles of the sum of the angles of any infinitely small triangle is proportional to its area. To give a tangible meaning to the curvature of an n-fold extended manifold at a given point, and in a given surface direction through it, we first mention that a shortest line emanating from a point is completely determined if its initial direction is given. Consequently we obtain a certain surface if we prolong all the initial directions from the given point which lie in the given surface element, into shortest lines; and this surface has a definite curvature at the given point, which is equal to the curvature of the n-fold extended manifold at the given point, in the given surface direction.

4.

Before applying these results to Space, it is still necessary to make some general considerations about flat manifolds, i.e., about manifolds in which the square of the line element can be represented as the sum of squares of complete differentials.

In a flat n-fold extended manifold the curvature in every direction, at every point, is zero; but according to the preceding investigation, in order to determine the metric relations it suffices to know that at each point the curvature is zero in $\frac{1}{2}n(n-1)$ independent surface-direction. The manifolds whose curvature is everywhere 0 can be considered as a special case of those manifolds whose curvature is everywhere constant. The common character of those manifolds whose curvature is constant may be expressed as follows: figures can be moved in them without stretching. For obviously figures could not be freely shifted and rotated in them if the curvature were not the same in all directions, at all points. On the other hand, the metric properties of the manifold are completely determined by the curvature; they are therefore exactly the same in all the directions around any one point as in the directions around any other, and thus the same constructions can be effected starting from either; consequently, in the manifolds with constant curvature figures may be given any arbitrary position. The metric relations of these manifolds depend only on the value of the curvature, and it may be mentioned, as regards the analytic presentation, that if one denotes this value by α, then the expression for the line element can be put in the form

$$\frac{1}{1+\frac{\alpha}{4}\Sigma x^2}\sqrt{\Sigma dx^2} .$$

5.

The consideration of surfaces with constant curvature may serve for a geometric illustration. It is easy to see that the surfaces whose curvature is positive can always be rolled onto a sphere whose radius is the reciprocal of the curvature; but in order to survey the multiplicity of these surfaces, let one of them be given the shape of a sphere, and the others the shape of surfaces of rotation which touch it along the equator. The surfaces with greater curvature than the sphere will then touch the sphere from inside and take a form like the portion of the surface of a ring, which is situated away from the axis; they could be rolled upon zones of spheres with smaller radii, but would go round more than once. Surfaces with smaller positive curvature are obtained from spheres of larger radii by cutting out a portion bounded by two great semi-circles, and bringing together the cut-lines. The surface of curvature zero will be a cylinder standing on the equator; the surfaces with negative curvature will touch this cylinder from outside and be formed like the part of the surface of a ring which is situated near the axis. If one regards these surfaces as possible positions for pieces of surface moving in them, as Space is for bodies, then pieces of surface can be moved in all these surfaces without stretching. The surfaces with positive curvature can always be so formed that pieces of surface can even be moved arbitrarily without bending, namely as spherical surfaces, but those with negative curvature cannot. Aside from this independence of position for surface pieces, in surfaces with zero curvature there is also an independence of position for directions, which does not hold in the other surfaces.

III. Applications to Space

1.

Following these investigations into the determination of the metric relations of an n-fold extended quantity, the conditions may be given which are sufficient and necessary for determining the metric relations of Space, if we assume beforehand the independence of lines from configuration and the possibility of expressing the line element as the square root of a second order differential expression, and thus flatness in the smallest parts.

First, these conditions may be expressed by saying that the curvature at every point equals zero in three surface directions, and thus the metric relations of Space are implied if the sum of the angles of a triangle always equals two right angles.

But secondly, if one assumes with Euclid not only the existence of lines independently of configuration, but also of bodies, then it follows that the curvature is everywhere constant, and the angle sum in all triangles is determined if it is known in one.

In the third place, finally, instead of assuming the length of lines to be independent of place and direction, one might assume that their length and direction is independent of place. According to this conception, changes or differences in position are complex quantities expressible in three independent units.

2.

In the course of the previous considerations, the relations of extension or regionality were first distinguished from the metric relations, and it was found that different metric relations were conceivable along with the same relations of extension; then systems of simple metric specifications were sought by means of which the metric relations of Space are completely determined, and from which all theorems about it are a necessary consequence. It remains now to discuss the question how, to what degree, and to what extent these assumptions are borne out by experience. In this connection there is an essential difference between mere relations of extension and metric relations, in that among the first, where the possible cases form a discrete manifold, the declarations of experience are to sure never completely certain, but they are not inexact, while for the second, where the possible cases form a continuous manifold, every determination from experience always remains inexact — be the probability ever so great that it is nearly exact. This circumstance becomes important when these empirical determinations are extended beyond the limits of observation into the immeasurably large and the immeasurably small; for the latter may obviously become ever more inexact beyond the boundary of observation, but not so the former.

When constructions in Space are extended into the immeasurably large, unboundedness is to be distinguished from infinitude; one belongs to relations of extension, the other to metric relations. That Space is an unbounded triply extended manifold is an assumption which is employed for every apprehension of the external world, by which at every moment the domain of actual perception is supplemented, and by which the possible locations of a sought for object are constructed; and in these applications it is continually confirmed. The unboundedness of space consequently has a greater empirical certainty than any experience of the external world. But its infinitude does not in any way follow from this; quite to the contrary, Space would necessarily be finite if one assumed independence of bodies from position, and thus ascribed to it a constant curvature, as long as this curvature had ever so small a positive value. If one prolonged the initial directions lying in a surface direction into shortest lines, one would obtain an unbounded surface with constant positive curvature, and thus a surface which in a flat triply extended manifold would take the form of a sphere, and consequently be finite.

3.

Questions about the immeasurably large are idle questions for the explanation of Nature. But the situation is quite different with questions about the immeasurably small. Upon the exactness with which we pursue phenomenon into the infinitely small, does our knowledge of their causal connections essentially depend. The progress of recent centuries in understanding the mechanisms of Nature depends almost entirely on the exactness of construction which has become possible through the invention of the analysis of the infinite and through the simple principles discovered by Archimedes, Galileo and Newton, which modern physics makes use of. By contrast, in the natural sciences where the simple principles for such constructions are still lacking, to discover causal connections one pursues phenomenon into the spatially small, just so far as the microscope permits. Questions about the metric relations of Space in the immeasurably small are thus not idle ones.

If one assumes that bodies exist independently of position, then the curvature is everywhere constant, and it then follows from astronomical measurements that it cannot be different from zero; or at any rate its reciprocal must be an area in comparison with which the range of our telescopes can be neglected. But if such an independence of bodies from position does not exist, then one cannot draw conclusions about metric relations in the infinitely small from those in the large; at every point the curvature can have arbitrary values in three directions, provided only that the total curvature of every measurable portion of Space is not perceptibly different from zero. Still more complicated relations can occur if the line element cannot be represented, as was presupposed, by the square root of a differential expression of the second degree. Now it seems that the empirical notions on which the metric determinations of Space are based, the concept of a solid body and that of a light ray, lose their validity in the infinitely small; it is therefore quite definitely conceivable that the metric relations of Space in the infinitely small do not conform to the hypotheses of geometry; and in fact one ought to assume this as soon as it permits a simpler way of explaining phenomena.

The question of the validity of the hypotheses of geometry in the infinitely small is connected with the question of the basis for the metric relations of space. In connection with this question, which may indeed still be ranked as part of the study of Space, the above remark is applicable, that in a discrete manifold the principle of metric relations is already contained in the concept of the manifold, but in a continuous one it must come from something else. Therefore, either the reality underlying Space must form a discrete manifold, or the basis for the metric relations must be sought outside it, in binding forces acting upon it.

An answer to these questions can be found only by starting from that conception of phenomena which has hitherto been approved by experience, for which Newton laid the foundation, and gradually modifying it under the compulsion of facts which cannot be explained by it. Investigations like the one just made, which begin from general concepts, can serve only to insure that this work is not hindered by too restricted concepts, and that progress in comprehending the connection of things is not obstructed by traditional prejudices.

This leads us away into the doamin of another science, the realm of physics, into which the nature of the present occasion does not allow us to enter.