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RELATIVISTIC FORCE TRANSFORMATION

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Abstract

Formulae relating one and the same force in two inertial frames of reference are derived directly from the Lorentz transformation of space and time coordinates and relativistic equation for the dynamic law of motion in three dimensions. We obtain firstly relativistic transformation for the velocity and acceleration of a particle. Then we substitute them in the relativistic dynamic equation and perform tedious algebraic manipulations. No recourse were made to "general rules for the transformation of 4-tensors". Formulae obtained were verified in electrodynamics.

1 Introduction

The relativistic mechanics looks in one dimension as [1]

$$\frac{\ddot{x}}{(1-\dot{x}^2/c^2)^{3/2}} = F \tag{1}$$

where $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$. Equation (1) is invariant under the Lorentz transformation

$$x' = \frac{x - vt}{(1-v^2/c^2)^{1/2}}, \tag{2}$$

$$y' = y, \quad z' = z, \tag{3}$$

$$t' = \frac{t - xv/c^2}{(1-v^2/c^2)^{1/2}} \tag{4}$$

where v is the parameter that has the meaning of the velocity

$$\mathbf{v} = (v, 0, 0) \tag{5}$$

which the inertial frame of reference K' moves in the inertial frame of reference K. Finding from (2), (4) relativistic transformations of the velocity $\dot{x}$ and acceleration $\ddot{x}$ of the body and substituting them in

$$\frac{\ddot{x}'}{(1-\dot{x}'^2/c^2)^{3/2}} = F' \tag{6}$$

we may verify that the left-hand part of (6) turns exactly into the left-hand part of (1). Hence, we have for the right-hand parts of equations (1) and (6)

$$F' = F. \tag{7}$$

In three dimensions the situation is complicated. The left-hand parts of scalar dynamic equations in K' are expressed as linear combinations of their left-hand parts in K. This induces respective transformation of the force $\mathbf{F}$. To find linear relations connecting each of F'_x, F'_y and F'_y with F_x, F_y and F_z is the aim of the present work. We will proceed in the following sequence.

Firstly, the one-dimensional Lorentz transformation (2)-(4) will be generalized to three dimensions. Then we will find from it the relativistic transformations of the velocity $\dot{\mathbf{r}} = d\mathbf{r}/dt$ and acceleration $\ddot{\mathbf{r}} = d^2\mathbf{r}/dt^2$, where $\mathbf{r} = (x, y, z)$. We will substitute them into a three-dimensional generalization of the dynamic equation (1) and after tedious manipulations find the relativistic transformation of $\mathbf{F}$. Finally, we will apply the result to the system of two electric charges moving with a constant velocity.

2 Three dimensional Lorentz transformation

Let $\mathbf{v}$ be arbitrarily oriented in space. We have from (2) and (4) for the projection of $\mathbf{r}$ on the direction of $\mathbf{v}$

$$\mathbf{r}' \cdot \mathbf{v}/v = \gamma(\mathbf{r} \cdot \mathbf{v}/v - vt), \tag{8}$$

$$t' = \gamma(t - \mathbf{r} \cdot \mathbf{v}/c^2) \tag{9}$$

where

$$\gamma = \frac{1}{(1 - v^2/c^2)^{1/2}}. \tag{10}$$

By (3) the direction perpendicular to $\mathbf{v}$ remains unchanged:

$$\mathbf{r}'_\perp = \mathbf{r}_\perp. \tag{11}$$

Expanding a vector into the sum of vectors perpendicular and parallel to $\mathbf{v}$ we get

$$\mathbf{r} = \mathbf{r}_\perp + \mathbf{r}_\parallel. \tag{12}$$

This gives, using (8), (11) and (12)

$$\begin{aligned} \mathbf{r}' &= \mathbf{r}'_\perp + \mathbf{r}'_\parallel = \mathbf{r}'_\perp + (\mathbf{r}' \cdot \mathbf{v}/v)\mathbf{v}/v \\ &= \mathbf{r}_\perp + \gamma(\mathbf{r} \cdot \mathbf{v}/v - vt)\mathbf{v}/v \\ &= \mathbf{r} + (\gamma - 1)\mathbf{r}_\parallel - \gamma \mathbf{v} t. \end{aligned} \tag{13}$$

3 Transformation of velocity

We have from (9) and (13)

$$dt' = \gamma(dt - d\mathbf{r} \cdot \mathbf{v}/c^2), \tag{14}$$

$$d\mathbf{r}' = d\mathbf{r} + (\gamma - 1)d\mathbf{r}_\parallel - \gamma \mathbf{v} dt. \tag{15}$$

We find from (14) and (15)

$$\dot{\mathbf{r}}' = \frac{d\mathbf{r}'}{dt'} = \frac{\dot{\mathbf{r}} + (\gamma - 1)\dot{\mathbf{r}}_\parallel - \gamma \mathbf{v}}{\gamma(1 - \dot{\mathbf{r}} \cdot \mathbf{v}/c^2)} = \frac{\dot{\mathbf{r}} + \mathbf{v}[(\gamma - 1)\dot{\mathbf{r}} \cdot \mathbf{v}/v^2 - \gamma]}{\gamma(1 - \dot{\mathbf{r}} \cdot \mathbf{v}/c^2)}. \quad (16)$$

If $\mathbf{v}$ is directed along the x-axis then we may get from (16) and (5)

$$\dot{x}' = \frac{\dot{x} - v}{1 - \dot{x}v/c^2}, \quad (17)$$

$$\dot{y}' = \frac{\dot{y}}{\gamma(1 - \dot{x}v/c^2)}, \quad \dot{z}' = \frac{\dot{z}}{\gamma(1 - \dot{x}v/c^2)}. \quad (18)$$

The following useful relation can be obtained from (17) and (18)

$$\frac{1}{(1 - \dot{\mathbf{r}}'^2/c^2)^{1/2}} = \frac{\gamma(1 - \dot{x}v/c^2)}{(1 - \dot{\mathbf{r}}^2/c^2)^{1/2}} \quad (19)$$

where

$$\dot{\mathbf{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2. \quad (20)$$

4 Transformation of acceleration

We have from (14) for the case of (5)

$$dt' = \gamma(dt - dxv/c^2) = dt\gamma(1 - \dot{x}v/c^2). \quad (21)$$

Differentiating (17) and using it and (21) we get

$$\ddot{x}' = \frac{d\dot{x}'}{dt'} = \frac{d\dot{x}'}{dt}\frac{dt}{dt'} = \left[\frac{\ddot{x}}{1 - \dot{x}v/c^2} + \frac{(\dot{x} - v)\ddot{x}v/c^2}{(1 - \dot{x}v/c^2)^2}\right]\frac{1}{\gamma(1 - \dot{x}v/c^2)}. \quad (22)$$

Using (10) in (22) gives finally

$$\ddot{x}' = \frac{\ddot{x}}{[\gamma(1 - \dot{x}v/c^2)]^3}. \quad (23)$$

Differentiating (18) and using it and (21) we get for a transverse acceleration

$$\ddot{y}' = \frac{d\dot{y}'}{dt'} = \frac{d\dot{y}'}{dt}\frac{dt}{dt'} = \gamma^{-1}\left[\frac{\ddot{y}}{1 - \dot{x}v/c^2} + \frac{\dot{y}\ddot{x}v/c^2}{(1 - \dot{x}v/c^2)^2}\right]\frac{1}{\gamma(1 - \dot{x}v/c^2)}. \quad (24)$$

Relation (24) gives

$$\ddot{y}' = \frac{1}{\gamma^2(1-\dot{x}v/c^2)^2}(\ddot{y} + \ddot{x}\frac{\dot{y}v/c^2}{1-\dot{x}v/c^2}). \quad (25)$$

The analogous expression for z is

$$\ddot{z}' = \frac{1}{\gamma^2(1-\dot{x}v/c^2)^2}(\ddot{z} + \ddot{x}\frac{\dot{z}v/c^2}{1-\dot{x}v/c^2}). \quad (26)$$

5 Transformation of $\mathbf{F}_{\|}$

The three-dimensional relativistic mechanics is [1]

$$\frac{d}{dt}\left[\frac{m\dot{\mathbf{r}}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}\right] = \mathbf{F}. \quad (27)$$

Completing the differentiation in (27) and taking scalar components:

$$\frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}} + \frac{m\dot{x}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{3/2}} = F_x, \quad (28)$$

$$\frac{m\ddot{y}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}} + \frac{m\dot{y}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{3/2}} = F_y, \quad (29)$$

$$\frac{m\ddot{z}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}} + \frac{m\dot{z}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{3/2}} = F_z. \quad (30)$$

Strictly speaking, equation (27) is not Lorentz invariant. However, we may retain the form of (28) in K' system:

$$\frac{m\ddot{x}'}{(1-\dot{\mathbf{r}}'^2/c^2)^{1/2}} + \frac{m\dot{x}'(\dot{\mathbf{r}}'\cdot\ddot{\mathbf{r}}')/c^2}{(1-\dot{\mathbf{r}}'^2/c^2)^{3/2}} = F'_x. \quad (31)$$

Substituting (17), (18), (23), (25) and (26) in (31), the left-hand part of (31) can be represented as a linear combination of left-hand parts of equations (28), (29) and (30). This means that retaining the form of (27) we must transform the right-hand part of (27). The component F'_x of the force is represented as respective linear combination of F_x, F_y and F_z. Next, we will perform explicitly the procedure mentioned.

Using (19) and (23) in (31) gives

$$\begin{aligned} F'_x = &\frac{m\ddot{x}}{\gamma^2(1-\dot{x}v/c^2)^2(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+ \\ &+\frac{m\dot{x}'(\dot{x}'\ddot{x}'+\dot{y}'\ddot{y}'+\dot{z}'\ddot{z}')\gamma^3(1-\dot{x}v/c^2)^3}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}. \end{aligned} \tag{32}$$

Then, substituting (23), (25), (26) and (17)-(18) in the second term of (32):

$$\begin{aligned} F'_x &= \frac{m\ddot{x}}{\gamma^2(1-\dot{x}v/c^2)^2(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m(\dot{x}-v)}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\left[\frac{(\dot{x}-v)\ddot{x}}{(1-\dot{x}v/c^2)^2}+\right. \\ &+ \left.\frac{\dot{y}}{1-\dot{x}v/c^2}(\ddot{y}+\ddot{x}\frac{\dot{y}v/c^2}{1-\dot{x}v/c^2})+\frac{\dot{z}}{1-\dot{x}v/c^2}(\ddot{z}+\ddot{x}\frac{\dot{z}v/c^2}{1-\dot{x}v/c^2})\right] \\ &= \frac{m\ddot{x}}{\gamma^2(1-\dot{x}v/c^2)^2(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m(\dot{x}-v)}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)}\times \\ &\times \left[\ddot{x}\frac{\dot{x}-v}{1-\dot{x}v/c^2}+\dot{y}\ddot{y}+\ddot{x}\frac{\dot{y}^2v/c^2}{1-\dot{x}v/c^2}+\dot{z}\ddot{z}+\ddot{x}\frac{\dot{z}^2v/c^2}{1-\dot{x}v/c^2}\right]. \end{aligned} \tag{33}$$

Firstly, we consider the portion of (33) that contains $\ddot{x}$. Using in it (10) and (20) gives

$$\begin{aligned} &\frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}(1-\dot{x}v/c^2)^2}\left[1-v^2/c^2+\right. \\ &\qquad+\left.\frac{(\dot{x}-v)^2+(\dot{y}^2+\dot{z}^2)(\dot{x}-v)v/c^2}{c^2(1-\dot{\mathbf{r}}^2/c^2)}\right]. \end{aligned} \tag{34}$$

The expression in quadratic brackets of (34) is

$$\begin{aligned} 1-v^2/c^2&+\frac{(\dot{x}-v)^2-(1-\dot{\mathbf{r}}^2/c^2)(\dot{x}-v)v+(\dot{x}-v)v-\dot{x}^2(\dot{x}-v)v/c^2}{c^2(1-\dot{\mathbf{r}}^2/c^2)} \\ &=1-\dot{x}v/c^2+\frac{\dot{x}^2-\dot{x}v-\dot{x}^2(\dot{x}-v)v/c^2}{c^2(1-\dot{\mathbf{r}}^2/c^2)} \\ &=1-\dot{x}v/c^2+\frac{\dot{x}(\dot{x}-v)(1-\dot{x}v/c^2)}{c^2(1-\dot{\mathbf{r}}^2/c^2)}. \end{aligned} \tag{35}$$

Substituting (35) in (34):

$$\begin{aligned} & \frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}(1-\dot{x}v/c^2)}\left[1+\frac{\dot{x}(\dot{x}-v)}{c^2(1-\dot{\mathbf{r}}^2/c^2)}\right] \\ = & \frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}(1-\dot{x}v/c^2)}\left[\frac{\dot{x}v}{c^2}+\frac{\dot{x}(\dot{x}-v)}{c^2(1-\dot{\mathbf{r}}^2/c^2)}\right] \\ = & \frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m\ddot{x}\dot{x}(\dot{x}-\dot{\mathbf{r}}^2v/c^2)}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)}. \end{aligned} \tag{36}$$

We have for members from (33) containing $\ddot{y}$ and $\ddot{z}$

$$\frac{m(\dot{x}-v)}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)}(\dot{y}\ddot{y}+\dot{z}\ddot{z}). \tag{37}$$

Summing (36) and (37) and using $\dot{x}\ddot{x}+\dot{y}\ddot{y}+\dot{z}\ddot{z}=\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}}$ gives

$$\begin{aligned} F'_x &= \frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m[\ddot{x}\dot{x}(\dot{x}-\dot{\mathbf{r}}^2v/c^2)+(\dot{x}-v)(\dot{y}\ddot{y}+\dot{z}\ddot{z})]}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)} \\ &= \frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m[\dot{x}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})-v(\dot{x}\ddot{x}\dot{\mathbf{r}}^2/c^2+\dot{y}\ddot{y}+\dot{z}\ddot{z})]}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)} \\ &= \frac{m\ddot{x}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m[\dot{x}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})-v(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})+v\dot{x}\ddot{x}(1-\dot{\mathbf{r}}^2/c^2)]}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)}. \end{aligned} \tag{38}$$

Using (28) in (38):

$$\begin{aligned} F'_x &= F_x+\frac{m[\dot{x}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})\dot{x}v/c^2-v(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})+v\dot{x}\ddot{x}(1-\dot{\mathbf{r}}^2/c^2)]}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)} \\ &= F_x+\frac{mv[-(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})(1-\dot{x}^2/c^2)+\dot{x}\ddot{x}(1-\dot{\mathbf{r}}^2/c^2)]}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)}. \end{aligned} \tag{39}$$

Using (20) in (39):

$$\begin{aligned} F'_x &= F_x-\frac{mv[(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})(\dot{y}^2+\dot{z}^2)/c^2+(\dot{y}\ddot{y}+\dot{z}\ddot{z})(1-\dot{\mathbf{r}}^2/c^2)]}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}(1-\dot{x}v/c^2)} \\ &= F_x-\frac{v/c^2}{(1-\dot{x}v/c^2)}\left\{\left[\frac{m\ddot{y}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m\dot{y}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\right]\dot{y}+\right. \\ &+ \left.\left[\frac{m\ddot{z}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}+\frac{m\dot{z}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\right]\dot{z}\right\}. \end{aligned} \tag{40}$$

Using (29) and (30) in (40) we get finally

$$F'_x = F_x - (F_y\dot{y} + F_z\dot{z})\frac{v/c^2}{(1-\dot{x}v/c^2)}. \tag{41}$$

6 Transformation of $F_\perp$

Using (18), (19), (25) and (17), (23), (26) in

$$\frac{m\ddot{y}'}{(1-\dot{\mathbf{r}}'^2/c^2)^{1/2}} + \frac{m\dot{y}'(\dot{\mathbf{r}}'\cdot\ddot{\mathbf{r}}')/c^2}{(1-\dot{\mathbf{r}}'^2/c^2)^{3/2}} = F'_y \tag{42}$$

we obtain

$$\begin{aligned}
F'_y &= \frac{m}{\gamma(1-\dot{x}v/c^2)}\{\frac{\ddot{y}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}} + \frac{\dot{y}\ddot{x}v/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}(1-\dot{x}v/c^2)} \\
&+ \frac{\dot{y}}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\left[\frac{(\dot{x}-v)\ddot{x}}{1-\dot{x}v/c^2} + \dot{y}\ddot{y} + \frac{\ddot{x}\dot{y}^2v/c^2}{1-\dot{x}v/c^2} + \dot{z}\ddot{z} + \frac{\ddot{x}\dot{z}^2v/c^2}{1-\dot{x}v/c^2}\right]\} \\
&= \frac{m}{\gamma(1-\dot{x}v/c^2)}\{\frac{\ddot{y}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}}\frac{\dot{y}}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\times \\
&\times \left[\frac{(1-\dot{\mathbf{r}}^2/c^2)\ddot{x}v + (\dot{x}-v)\ddot{x} + (\dot{y}^2+\dot{z}^2)\ddot{x}v/c^2}{1-\dot{x}v/c^2} + \dot{y}\ddot{y} + +\dot{z}\ddot{z}\right]\} \\
&= \frac{m}{\gamma(1-\dot{x}v/c^2)}\left\{\frac{\ddot{y}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}} + \frac{\dot{y}}{c^2(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\times\right. \\
&\times \left.\left[\frac{(-\dot{x}^2/c^2)\ddot{x}v + \dot{x}\ddot{x}}{1-\dot{x}v/c^2} + \dot{y}\ddot{y} + +\dot{z}\ddot{z}\right]\right\} \\
&= \frac{m}{\gamma(1-\dot{x}v/c^2)}\left\{\frac{\ddot{y}}{(1-\dot{\mathbf{r}}^2/c^2)^{1/2}} + \frac{\dot{y}(\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}})/c^2}{(1-\dot{\mathbf{r}}^2/c^2)^{3/2}}\right\}.
\end{aligned} \tag{43}$$

Comparing (43) with (29) and using in it (10) we get finally

$$F'_y = F_y\frac{(1-v^2/c^2)^{1/2}}{1-\dot{x}v/c^2}. \tag{44}$$

Similarly for z component:

$$F'_z = F_z\frac{(1-v^2/c^2)^{1/2}}{1-\dot{x}v/c^2}. \tag{45}$$

7 Relativistic electrodynamics

Let two particles at $(0,0,0)$ and (x,y,z) be at rest in the reference system K. They interact with a force $\breve{\mathbf{F}}$ that can be calculated from some field equations. Next, let these particles move with a constant velocity

$$\dot{\mathbf{r}} = (\dot{x}, 0, 0). \tag{46}$$

We may calculate the force $\mathbf{F}$ acted between moving particles from the same field equations. A force can be expanded into the sum of longitudinal and transverse components:

$$\mathbf{F} = \mathbf{F}_{\|} + \mathbf{F}_{\perp}. \tag{47}$$

Let us pass to the reference system K' given by (5) with

$$v = \dot{x}. \tag{48}$$

Then, according to (41) with (46):

$$\mathbf{F}'_{\|} = \mathbf{F}_{\|}, \tag{49}$$

according to (45) with (48) and (10):

$$\mathbf{F}'_{\perp} = \gamma \mathbf{F}_{\perp}. \tag{50}$$

By (47), (49) and (50):

$$\mathbf{F}' = \mathbf{F}'_{\|} + \mathbf{F}'_{\perp} = \mathbf{F}_{\|} + \gamma \mathbf{F}_{\perp}. \tag{51}$$

The principle of relativity states that we must have

$$\mathbf{F}' = \breve{\mathbf{F}} \tag{52}$$

when $x' = x$, $y' = y$ and $z' = z$. Further we will verify (52) for the case of two electric charges.

We have for two charges q_1 and q_2 at rest

$$\breve{\mathbf{F}} = q_1 q_2 \frac{x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z}{(x^2+y^2+z^2)^{3/2}}. \tag{53}$$

When a charge q_1 moves with a constant velocity $\dot{x}$ we must solve equations

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi q_1\delta(x-\dot{x}t,y,z), \quad (54)$$

$$\frac{\partial^2 A_x}{\partial x^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 A_x}{\partial t^2} = -\frac{4\pi\dot{x}}{c}q_1\delta(x-\dot{x}t,y,z) \quad (55)$$

Using the Lorentz transform (2)-(4) with (48) we may pass in (54) and (55) to reference system K'. The left-hand parts of equations (54) and (55) are known to be Lorentz-invariant. In K' the charge is at rest, hence fields φ and $\mathbf{A}$ do not depend on t'. Using the property of δ-function $\delta(|a|x)=\delta(x)/|a|$ we obtain from (54) and (55)

$$\frac{\partial^2\varphi}{\partial x'^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2} = -4\pi q_1\gamma\delta(x',y,z), \quad (56)$$

$$\frac{\partial^2 A_x}{\partial x'^2}+\frac{\partial^2 A_x}{\partial y^2}+\frac{\partial^2 A_x}{\partial z^2} = -4\pi q_1\gamma\frac{\dot{x}}{c}\delta(x',y,z). \quad (57)$$

Solving equations (56) and (57) we get with (48), (2) and (10)

$$\varphi = \gamma\frac{q_1}{R}, \quad (58)$$

$$A_x = \gamma\frac{v}{c}\frac{q_1}{R}, \quad A_y=0, \quad A_z=0, \quad (59)$$

$$R = [\gamma(x-vt)^2+y^2+z^2]^{1/2}. \quad (60)$$

Calculating from (58)-(60) the Lorentz force that acts on a charge q_2 which moves in K with the same velocity $\mathbf{v}$ we may obtain[2]

$$\begin{aligned}\mathbf{F} &= q_2[-\boldsymbol{\nabla}\varphi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\frac{1}{c}(\mathbf{v}\times\mathtt{curl}\mathbf{A})] \\ &= q_1q_2\frac{\gamma(x-vt)\mathbf{i}_x+\gamma^{-1}(y\mathbf{i}_y+z\mathbf{i}_z)}{[\gamma^2(x-vt)^2+y^2+z^2]^{3/2}}. \quad (61)\end{aligned}$$

We may isolate in (61) longitudinal $\mathbf{F}_\parallel = F_x\mathbf{i}_x$ and transverse $\mathbf{F}_\perp = F_y\mathbf{i}_y+F_z\mathbf{i}_z$ components according to (47), then substitute them into (51) and use (2) with (10) in the result. This gives

$$\mathbf{F}' = q_1q_2\frac{x'\mathbf{i}_x+y\mathbf{i}_y+z\mathbf{i}_z}{(x'^2+y^2+z^2)^{3/2}}. \quad (62)$$

Comparing (62) and (53) for $x=x'$ we confirm[2] formula (52).

References

[1] H.Goldstein, *Classical Mechanics*, Addison-Wesley, Reading, Massachusetts, 1980.

[2] Valery P. Dmitriyev, "The easiest way to the Heaviside ellipsoid," Am. J. Phys. **70**, 717-718 (2002).