Bull. Korean Math. Soc. **31** (1994), No. 2, pp. 201–214

# DEPTHS OF THE REES ALGEBRAS AND THE ASSOCIATED GRADED RINGS

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## 1. Introduction

Throughout this paper, all rings are assumed to be commutative with identity. By a local ring $(R, m)$, we mean a Noetherian ring $R$ which has a unique maximal ideal $m$. By $\dim(R)$ we always mean the Krull dimension of $R$. Let $I$ be an ideal in a ring $R$ and $t$ an indeterminate over $R$. Then the Rees algebra $R[It]$ and the associated graded ring $gr_I(R)$ of $I$ are defined to be

$$R[It] = R \oplus It \oplus I^2t^2 \oplus \cdots$$

and

$$gr_I(R) = R/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$$

These rings are important not only algebraically, but geometrically as well. For example, $\mathrm{Proj}(R[It])$ is the blow-up of $\mathrm{Spec}(R)$ with respect to $I$.

The purpose of this paper is to investigate the relationship between the depths of the Rees algebra $R[It]$ and the associated graded ring $gr_I(R)$ of an ideal $I$ in a local ring $(R, m)$ of $\dim(R) > 0$. The relationship between the Cohen-Macaulayness of these two rings has been studied extensively. Let $(R, m)$ be a local ring and $I$ an ideal of $R$. An ideal $J$ contained in $I$ is called a <u>reduction of $I$</u> if $JI^n = I^{n+1}$ for some integer $n \geq 0$. A reduction $J$ of $I$ is called a <u>minimal reduction of $I$</u> if $J$ is minimal with respect to being a reduction of $I$. The <u>reduction number of $I$</u> <u>with respect to $J$</u> is defined by

$$r_J(I) = \min\{n \geq 0 \mid JI^n = I^{n+1}\}.$$

Received April 15, 1993.

The reduction number of $I$ is defined by

$$r(I) = \min\{r_J(I) \mid J \quad \text{is a minimal reduction of } I\}.$$

S. Goto and Y. Shimoda characterized the Cohen-Macaulay property of the Rees algebra of the maximal ideal of a Cohen-Macaulay local ring in terms of the Cohen-Macaulay property of the associated graded ring of that maximal ideal and the reduction number of that maximal ideal. Let us state their theorem.

THEOREM 1.1. *([4], Theorem 3.1) Let $(R, m)$ be a Cohen-Macaulay local ring of dimension $d > 0$ and assume that $R/m$ is infinite. Then the following conditions are equivalent.*
*(1) $R[mt]$ is a Cohen-Macaulay ring.*
*(2) $gr_m(R)$ is a Cohen-Macaulay ring and $r(m) \leq d - 1$.*

In a number of cases, this theorem gives a test for determining whether or not $R[mt]$ is Cohen-Macaulay, because $r(m)$ is reasonable to compute. For example, let $R = k[[X^2, X^3]]$ and $m = (X^2, X^3)R$, where $k$ is a field and $X$ is variable over $k$. Then $R$ is one-dimensional local domain and $r(m) = 1$. Hence $R[mt]$ is not Cohen-Macaulay by Theorem 1.1. More generally, if $(R, m)$ is any one-dimensional local domain which is not a rank one discrete valuation domain, then $R[mt]$ is not Cohen-Macaulay by Theorem 1.1.

Let $(R, m)$ be a local ring and $I$ an ideal of $R$. The analytic spread of $I$, denoted by $l(I)$, is defined to be $\dim(R[It]/mR[It])$. In [13], it is shown that $ht(I) \leq l(I) \leq \dim(R)$. An ideal $I$ is called equimultiple if $l(I) = ht(I)$. If $R/m$ is an infinite field, then $l(I)$ is the least number of elements generating a reduction of $I$ ([13]). In particular, all $m$-primary ideals are equimultiple. U. Grothe, M. Herrmann and U. Orbanz generalized Theorem 1.1 to the case of all “equimultiple ideals". We now state the result of Grothe - Herrmann - Orbanz.

THEOREM 1.2. *([5], Theorem 4.8) Let $(R, m)$ be a Cohen-Macaulay local ring having an infinite residue field and $I$ an equimultiple ideal of height $s$. Assume that $s > 0$. Then the following conditions are equivalent.*
*(1) $R[It]$ is a Cohen-Macaulay ring.*
*(2) $gr_I(R)$ is a Cohen-Macaulay ring and $r(I) \leq s - 1$.*

In general, it is known (*cf.* [9], Proposition 1.1) that if $R$ and $R[It]$ are Cohen-Macaulay, then $\text{depth}(R[It]) = \text{depth}(gr_I(R)) + 1$. On the other hand, if $gr_I(R)$ is Cohen-Macaulay, then $\text{depth}(R[It]) \leq 1 + \text{depth}(gr_I(R))$ (see Lemma 3.1). We shall prove that the following equality

$$\text{depth}(R[It]) = \text{depth}(gr_I(R)) + 1$$

always holds for ideal $I$ under negation of the Cohen-Macaulay assumption on $gr_I(R)$ and the condition that $R$ is normally Cohen-Macaulay along $I$. We also characterize that the property of Cohen-Macaulayness of $R[It]$ and $gr_I(R)$ are equivalent for an equimultiple ideal $I$ by imposing the condition of a regular local ring on $R$. As a general reference, we refer the reader to [11] for any unexplained notation and terminology.

## 2. Preliminaries

Let $R$ be a Noetherian ring and $I$ an ideal of $R$. Given an element $a \in R$, we define

$$v_I(a) = \begin{cases} n & \text{if } a \in I^n \backslash I^{n+1} \\ \infty & \text{if } a \in \cap_{n \geq 1} I^n. \end{cases}$$

When $v_I(a) = n \neq \infty$, the residue class of $a$ in $I^n/I^{n+1}$ is called the leading form of $a$ and denoted by $a^*$. If $v_I(a) = \infty$, then we set $a^* = 0$.

LEMMA 2.1. *Let $R$ be a Noetherian ring and $I$ an ideal in $R$. Let $n$ be a non-negative integer and $b \in R$. Assume that $bR \cap I^i = bI^{i-n}$ for $i \geq n$. Let $R_1 = R/bR$ and $I_1 = IR_1$. Then*

$$R_1[I_1t] \cong \frac{R[It]}{(b, bt, \cdots, bt^n)}$$

*as graded $R$-algebras.*

*Proof.* : Note that $bR \cap I^j = bR$ for $j < n$. Let $\phi : R[It] \longrightarrow R_1[I_1t]$ denote the canonical epimorphism. Put $K = \text{Ker}\phi$. Then $K$ is a homogeneous ideal in $R[It]$.

**Claim** : $K = (b, bt, \cdots, bt^n)$.

$\supseteq$ : It is obvious.

$\subseteq$ : Let $z$ be a homogeneous element of $K$ with $\deg z = l \geq 0$. Write $z = \alpha t^l$ with $\alpha \in I^l$. Then we have $\alpha \in bR \cap I^l$. We have two cases : (1) when $l \geq n$, and (2) when $l < n$.
Case (1) : $l \geq n$. By assumption we write $\alpha = bc$ with $c \in I^{l-n}$, and hence

$$z = \alpha t^l = bct^l = bt^n \cdot ct^{l-n} \in (bt^n)R[It].$$

Case (2) : $l < n$. From the note, we write $\alpha = br$ with $r \in R$, and hence

$$z = \alpha t^l = rbt^l \in (bt^l)R[It]. \qquad \blacksquare$$

LEMMA 2.2. *Let $R$ be a Noetherian ring, $I$ an ideal in $R$ and $a \in R$. Assume that $a$ is a non-zero-divisior on $R$ and $aR \cap I^n = aI^{n-1}$ for $n \geq 1$. Then*
*(1)* $\big(aR[It] : at\big) = IR[It]$.
*(2) There exists an exact sequence*

$$0 \longrightarrow gr_I(R) \longrightarrow \frac{R[It]}{aR[It]} \longrightarrow \left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right] \longrightarrow 0$$

*of graded $R[It]$-modules.*

*Proof.* : (1) $\supseteq$ : Let $f \in IR[It]$. Write $f = f_0 + f_1 t + \cdots + f_s t^s$, where $f_i \in I^{i+1}$, $i = 0, 1, \cdots, s$. Then we have

$$f \cdot at = a(f_0 t + f_1 t^2 + \cdots + f_s t^{s+1}) \in aR[It].$$

$\subseteq$ : Let $f \in \big(aR[It] : at\big)$ with $f = f_0 + f_1 t + \cdots + f_l t^l \in R[It]$. Then $f \cdot at = ag$, where $g = g_0 + g_1 t + \cdots + g_{l+1} t^{l+1} \in R[It]$. Hence we have

$$ag_0 + (ag_1 - af_0)t + \cdots + (ag_{l+1} - af_l)t^{l+1} = 0.$$

By the nature of $a$, $f_i = g_{i+1} \in I^{i+1}$ for $i = 0, 1, \cdots, l$, which concludes the proof of (1).
(2) Consider the exact sequence

$$0 \longrightarrow \frac{(a, at)R[It]}{(a)R[It]} \longrightarrow \frac{R[It]}{aR[It]} \longrightarrow \frac{R[It]}{(a, at)R[It]} \longrightarrow 0$$

of graded $R[It]$-modules. Moreover

$$\begin{aligned}\frac{(a,at)R[It]}{aR[It]} &\cong \frac{(at)R[It]}{aR[It]\cap(at)R[It]} = \frac{(at)R[It]}{(aR[It]:at)(at)}\\ &\cong \frac{R[It]}{(aR[It]:at)} = \frac{R[It]}{IR[It]} \qquad \text{by (1)}\\ &\cong gr_I(R),\end{aligned}$$

and

$$\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right] \cong \frac{R[It]}{(a,at)R[It]} \qquad \text{by Lemma 2.1} \qquad \blacksquare$$

**Notation** : Let $G = \oplus_{n\geq 0}G_n$ be a non-negatively graded Noetherian ring such that $G_0$ is a local ring and $A$ a finitely generated graded $G$-module. Then we define depth$(A)$ to be depth$_{G_N}(A_N)$, where $N$ is the unique homogeneous maximal ideal of $G$. We let $G^+$ denote the ideal $\oplus_{n\geq 1}G_n$.

LEMMA 2.3. *( cf. [3], Lemma 1.1) Let $G$ be a non-negatively graded Noetherian ring such that $G_0$ is a local ring and $A, B$ and $C$ be finitely generated graded $G$-modules. Suppose there is an exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

*where the maps are all homogeneous. Then either*
*(1) $\text{depth}A \geq \text{depth}B = \text{depth}C$, or*
*(2) $\text{depth}B \geq \text{depth}A = \text{depth}C + 1$, or*
*(3) $\text{depth}C > \text{depth}A = \text{depth}B$.*

*Proof.* : The proof follows from the Ext characterization of depth, and the long exact sequence for Ext. $\blacksquare$

DEFINITION 2.4. Let $(R, m)$ be a local ring and $I$ an ideal of $R$. We say $R$ is <u>normally Cohen-Macaulay along $I$</u> if

$$\text{depth}\left(I^n/I^{n+1}\right) = \dim(R/I) \qquad \text{for all } n \geq 0.$$

REMARKS. : (1) Let $(R, m)$ be a local ring. Then $R$ is normally Cohen-Macaulay along any $m$-primary ideal $I$.
(2) Let $(R, m)$ be a quasi-unmixed local ring and $I$ an ideal in $R$ with $ht(I) > 0$. Assume that $R$ is normally Cohen-Macaulay along $I$. Then $I$ is an equimultiple ideal.
(3) Let $(R, m)$ be a local ring and $I$ an ideal of $R$, and suppose that $R$ is normally Cohen-Macaulay along $I$. Suppose that $b^*$, the image of $b$ in $R/I$, is a $gr_I(R)$-regular element. Then $R/bR$ is normally Cohen-Macaulay along $I(R/bR)$.

*Proof.* : (1) It is trivial.
(2) Recall that $\dim(R) = \dim(R/I) + ht(I)$ since $R$ is a quasi-unmixed local ring. $R/I^n$ is Cohen-Macaulay for all $n \geq 1$ ([6], Lemma 3.8). Then we have by a result of L. Burch ([1], Corollary in pp. 373)

$$\begin{aligned} l(I) &\leq \dim(R) - \min_n\{\operatorname{depth}(R/I^n)\} \\ &= \dim(R) - \operatorname{depth}(R/I^{n_0}), \qquad \text{for some integer } n_0 \\ &= \dim(R) - \dim(R/I^{n_0}) \\ &= ht(I^{n_0}) \\ &= ht(I). \end{aligned}$$

(3) Put $R_1 = R/bR$ and $I_1 = IR_1$. We have the following isomorphisms

$$(I_1)^n/(I_1)^{n+1} \cong \frac{I^n + bR}{I^{n+1} + bR} \cong \frac{I^n}{I^{n+1} + bI^n} \cong \frac{I^n/I^{n+1}}{b(I^n/I^{n+1})}.$$

Since $b^*$ is a $gr_I(R)$-regular element, $b$ is a non-zero-divisor on $I^n/I^{n+1}$ for all $n \geq 0$. Hence, we have

$$\begin{aligned} \operatorname{depth}\left(I_1^n/I_1^{n+1}\right) &= \operatorname{depth}\left(I^n/I^{n+1}\right) - 1 \\ &= \dim(R/I) - 1 \\ &= \dim(R_1/I_1). \qquad \blacksquare \end{aligned}$$

## 3. Depths of the Rees algebras and the associated graded rings

LEMMA 3.1. *Let $(R, m)$ be a $d$-dimensional Cohen-Macaulay local ring and $I$ an ideal of $ht(I) > 0$. Then*

$$\textit{depth}(R[It]) \leq \textit{depth}(gr_I(R)) + 1.$$

*Proof.* : Consider the exact sequences

$$0 \longrightarrow ItR[It] \longrightarrow R[It] \longrightarrow R \longrightarrow 0 \tag{1}$$

$$0 \longrightarrow IR[It] \longrightarrow R[It] \longrightarrow gr_I(R) \longrightarrow 0 \tag{2}$$

of $R[It]$-modules. From (2) we have that by Lemma 2.3, either

$$\text{depth}(R[It]) \geq \text{depth}(IR[It]) = \text{depth}(gr_I(R)) + 1,$$

or

$$\text{depth}(gr_I(R)) \geq \text{depth}(R[It]).$$

In the second case we are done. Hence we assume that

$$\text{depth}(IR[It]) = \text{depth}(gr_I(R)) + 1. \tag{3}$$

From (1) it follows that by Lemma 2.3, either

$$\text{depth}(ItR[It]) \geq \text{depth}(R[It]),$$

or

$$\text{depth}(R[It]) \geq \text{depth}(ItR[It]) = \text{depth}(R) + 1.$$

But since $IR[It] \cong ItR[It]$ as $R[It]$-modules, we have

$$\text{depth}(IR[It]) = \text{depth}(ItR[It]). \tag{4}$$

First, if $\text{depth}(ItR[It]) \geq \text{depth}(R[It])$, then

$$\begin{aligned} \text{depth}(gr_I(R)) + 1 &= \text{depth}(IR[It]) \qquad \text{by (3)} \\ &= \text{depth}(ItR[It]) \qquad \text{by (4)} \\ &\geq \text{depth}(R[It]). \end{aligned}$$

Second, if $\text{depth}(ItR[It]) = \text{depth}(R) + 1$, then

$$\begin{aligned} \text{depth}(gr_I(R)) + 1 &= \text{depth}(IR[It]) \qquad \text{by (3)} \\ &= \text{depth}(ItR[It]) \qquad \text{by (4)} \\ &= \text{depth}(R) + 1 \\ &= \dim(R) + 1 \qquad (\text{R : CML}) \\ &= \dim(R[It]) \\ &\geq \text{depth}(R[It]). \end{aligned}$$

Thus, in all cases we have

$$\text{depth}(R[It]) \leq \text{depth}(gr_I(R)) + 1. \qquad \blacksquare$$

LEMMA 3.2. *Let $V$ be a finite-dimensional vector space over the infinite field $K$, and let $H_1, \cdots, H_n$ be proper subspaces of $V$. Then there exists $v \in V$ such that $v \notin H_1 \cup \cdots \cup H_n$.*

*Proof.* : We proceed by induction on $n$. If $n = 1$, then it is clear. If $n > 1$, then we can choose an element $\alpha \in V$ such that $\alpha \notin H_1 \cup \cdots \cup H_{n-1}$ by inductive hypothesis. By the nature of $H_n$, there exists an element $\beta \in V \setminus H_n$. Suppose that $H_1 \cup \cdots \cup H_n = V$. Since $K$ is infinite, there are distinct elements $r_1, \cdots, r_{n+1}$ in $K$ such that $\alpha + r_1\beta, \cdots, \alpha + r_{n+1}\beta$ are in $V$. By the pigeonhole principle, two of them must be in the same subspace, say $\alpha + r_i\beta, \alpha + r_j\beta$ are in $H_k$ for some $k$, where $i \neq j$. If $k = n$, then $(\alpha + r_i\beta) - (\alpha + r_j\beta) = (r_i - r_j)\beta \in H_n$. Hence $\beta \in H_n$, which is a contradiction to the choice of $\beta$. If $k < n$, then $(r_i - r_j)\beta \in H_k$, and hence $\beta \in H_k$. Since $\alpha + r_i\beta \in H_k$, it follows that $\alpha \in H_k$, which is a contradiction to the choice of $\alpha$. $\blacksquare$

LEMMA 3.3. *Let $(R, m)$ be a local ring and $I$ an ideal in $R$ of $ht(I) > 0$. Suppose that*

$$\textit{depth}(I^n/I^{n+1}) > 0 \qquad \textit{for all} \quad n \geq 0.$$

*Then we can find an element $x \in m$ which is a non-zero-divisor on $R/I^n$ for all $n \geq 0$.*

*Proof.* : Since $\bigcup_n \mathrm{Ass}_{R/I}\left(I^n/I^{n+1}\right) \subseteq \mathrm{Ass}_{R/I}(gr_I(R))$ and $\mathrm{Ass}_{R/I}(gr_I(R))$ is a finite set (*cf*, [12], Proposition 1.3), and hence $\bigcup_n \mathrm{Ass}_{R/I}\left(I^n/I^{n+1}\right)$ is a finite set. We can choose an element $x \in m$ which is a non-zero-divisor on $I^n/I^{n+1}$ for all $n \geq 0$.

**Claim** : $x$ is a non-zero-divisor on $R/I^{n+1}$ for all $n \geq 0$.

This will be done by induction on $n$. The assertion is clear for $n = 0$. So we assume $n \geq 1$. Since $x$ is a non-zero-divisor on $I^n/I^{n+1}$ and on $R/I^n$, $x$ is a non-zero-divisor on $R/I^{n+1}$ by considering a short exact sequence. ∎

THEOREM 3.4. *Let $(R, m)$ be a positive integer $d$-dimensional Cohen-Macaulay local ring having an infinite residue field $k$ and $I$ an ideal with $ht(I) > 0$. Assume that $gr_I(R)$ is not Cohen-Macaulay and $R$ is normally Cohen-Macaulay along $I$. Then*

$$depth(R[It]) = depth(gr_I(R)) + 1.$$

*Proof.* : The inequality $\leq$ holds by Lemma 3.1. We now prove the other inequality. We proceed by induction on $r = \dim(R/I)$. We have two cases : (1) when $r = 0$, and (2) when $r > 0$.

Case (1) : $r = 0$. In this case $I$ is an $m$-primary ideal of $R$. We now proceed by induction on $d = \dim(R)$. Since the inequality is trivial if either $d = 1$ or $\mathrm{depth}(gr_I(R)) = 0$, we may assume that $d \geq 2$ and $\mathrm{depth}(gr_I(R)) \geq 1$. Since $I$ is an $m$-primary ideal of $R$, any homogeneous element of degree 0 that is not a unit is nilpotent in $gr_I(R)$. Hence there exists a regular element in $gr_I(R)^+$. That is, $gr_I(R)^+ \not\subseteq \bigcup\{\, Q \mid Q \in \mathrm{Ass}(gr_I(R))\,\}$. For each $Q \in \mathrm{Ass}(gr_I(R))$, $\left((Q \cap I/I^2) + mI/I^2\right)/(mI/I^2)$ is a proper $k$-vector subspace of $I/mI$ by Nakayama's Lemma. Since $k$ is infinite, we can choose $a \in I \backslash mI$ such that the image of $a$ in $I/I^2$, $a^*$, is not in any associated prime $Q$ of $gr_I(R)$ by Lemma 3.2. That is, $a^*$ is a $gr_I(R)$-regular element. Hence $a$ is a non-zero-divisor on $R$ and $aR \cap I^n = aI^{n-1}$ for all $n \geq 1$ (*cf* : [14], Corollary 2.7). We have an exact sequence

$$0 \longrightarrow gr_I(R) \longrightarrow \frac{R[It]}{aR[It]} \longrightarrow \left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right] \longrightarrow 0$$

of $R[It]$-modules by Lemma 2.2. Applying Lemma 2.3, we see that either

$$\mathrm{depth}(gr_I(R)) \geq \mathrm{depth}\left(\frac{R[It]}{(a)}\right) = \mathrm{depth}\left(\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right]\right),$$

or

$$\operatorname{depth}\left(\frac{R[It]}{(a)}\right) \geq \operatorname{depth}(gr_I(R)) = \operatorname{depth}\left(\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right]\right) + 1,$$

or

$$\operatorname{depth}\left(\left(\frac{R}{aR}\right)\left[\frac{I}{aR}t\right]\right) > \operatorname{depth}(gr_I(R)) = \operatorname{depth}\left(\frac{R[It]}{(a)}\right).$$

But as $a^*$ is a $gr_I(R)$-regular element, $gr_I(R)/(a^*) \cong gr_{I_1}(R_1)$, where $R_1 = R/aR$ and $I_1 = IR_1$. First, if depth $(R[It]/(a)) = \operatorname{depth}(R_1[I_1t])$, then

$$\begin{aligned}
\operatorname{depth}(R[It]) &= \operatorname{depth}\left(\frac{R[It]}{(a)}\right) + 1 \\
&= \operatorname{depth}(R_1[I_1t]) + 1 \\
&\geq \operatorname{depth}(gr_{I_1}(R_1)) + 1 + 1 \\
&= \operatorname{depth}\left(\frac{gr_I(R)}{(a^*)}\right) + 2 \\
&= \operatorname{depth}(gr_I(R)) - 1 + 2 \\
&= \operatorname{depth}(gr_I(R)) + 1.
\end{aligned}$$

Second, if $\operatorname{depth}(R[It]/(a)) \geq \operatorname{depth}(gr_I(R))$, then

$$\begin{aligned}
\operatorname{depth}(R[It]) &= \operatorname{depth}(R[It]/(a)) + 1 \\
&\geq \operatorname{depth}(gr_I(R)) + 1.
\end{aligned}$$

Third, if $\operatorname{depth}(gr_I(R)) = \operatorname{depth}\big(R[It]/(a)\big)$, then the assertion is clear. Thus, this completes the proof of case (1).

Case (2) : $r > 0$. Assume that the inequality holds for $r - 1$. Since $R$ is normally Cohen-Macaulay along $I$, we can choose an element $b \in m$ which is a regular element on $R/I^{n+1}$ for all $n \geq 0$ by Lemma 3.3, and hence $b$ is a non-zero-divisor on $R$ and $bR \cap I^n = bI^n$ for all $n \geq 1$ ($cf$: [6], Lemma 1.35). Applying Lemma 2.1, we get the following isomorphism $R[It]/(b) \cong R_2[I_2t]$, where $R_2 = R/bR$ and $I_2 = IR_2$. Hence $\dim\big(R_2/I_2\big) = \dim\big(R/(I, b)\big) =$

dim $\big(R/I\big) - 1$, and $gr_{I_2}(R_2) \cong gr_I(R)/(b^*)$ is not Cohen-Macaulay, as $b^*$ is a $gr_I(R)$-regular element and $R_2$ is normally Cohen-Macaulay along $I_2$. By the inductive hypothesis, we have

$$\text{depth}(R_2[I_2t]) \geq \text{depth}(gr_{I_2}(R_2)) + 1.$$
$$\text{depth}(R[It]) - 1 \geq \text{depth}(gr_I(R)) - 1 + 1.$$

This completes the proof of case (2). ∎

COROLLARY 3.4.1. *([8], Theorem2.1 ) Let $(R, m)$ be a Cohen-Maca ulay local ring of dimension $d \geq 1$ and $I$ an $m$-primary ideal. Assume that $gr_I(R)$ is not Cohen-Macaulay. Then*

$$\mathit{depth}(R[It]) = \mathit{depth}\big(gr_I(R)\big) + 1.$$

*Proof.* : Recall that $R$ is normally Cohen-Macaulay along any $m$-primary ideal. ∎

We next show that the property of Cohen-Macaulayness of $R[It]$ and $gr_I(R)$ are equivalent for equimultiple ideals by imposing the conditions of a $RLR$ (Regular Local Ring) on $R$. In other words, using a consequence of the Briançon - Skoda Theorem we can drop the condition $r(I) \leq s-1$ in Theorem 1.2. Recall that an element $a \in R$ is integral over an ideal $I$ if it satisfies an equation of the form

$$a^n + r_1a^{n-1} + \cdots + r_n = 0, \qquad r_i \in I^i.$$

The set of all elements which are integral over an ideal $I$ form an ideal, denoted by $\overline{I}$ and called the integral closure of $I$.

REMARKS. : (1) Let $R$ be a Noetherian ring. Then an ideal $J \subseteq I$ is a reduction of $I$ if and only if $I \subseteq \overline{J}$.
(2) The Briançon-Skoda Theorem (see [2], [10], or [7]) states that if $(R, m)$ is a regular local ring and $I$ is an ideal generated by $n$ elements, then $\overline{I^n} \subseteq I$.

LEMMA 3.5. *Let $(R, m)$ be a regular local ring with an infinite residue field and $I$ an equimultiple ideal with $ht(I) = s > 0$. Assume that $gr_I(R)$ is a Cohen-Macaulay ring. Then there exist elements $a_1, \cdots, a_s$ in $I$ such that $I^s = (a_1, \cdots, a_s)I^{s-1}$.*

*Proof.* : Let $(a_1, \cdots, a_s)$ be a minimal reduction of $I$. Let $b_1, \cdots, b_r$ be a system of parameters mod$I$, where $r = \dim(R/I) = \dim(R) - ht(I)$. Then $\{b_1^*, \cdots, b_r^*, a_1^*, \cdots, a_s^*\}$ is a homogeneous system of parameters for $gr_I(R)$, where $\deg b_i^* = 0$ for $i = 1, \cdots, r$, and $\deg a_j^* = 1$ for $j = 1, \cdots, s$ ($cf$: [5],Corollary 2.7). Hence it is a $gr_I(R)$-regular sequence since $gr_I(R)$ is Cohen-Macaulay. We have $(a_1, \cdots, a_s) \bigcap I^n = (a_1, \cdots, a_s)I^{n-1}$, $\forall n \geq 1$ ($cf$: [14],Corollary 2.7). $(a_1, \cdots, a_s)^s$ is a reduction of $I^s$ since $(a_1, \cdots, a_s)$ is a reduction of $I$. Then

$$(a_1, \cdots, a_s)^s \subseteq I^s \subseteq \overline{(a_1, \cdots, a_s)^s} \subseteq (a_1, \cdots, a_s).$$

Hence we have

$$(a_1, \cdots, a_s)I^{s-1} = (a_1, \cdots, a_s) \bigcap I^s = I^s. \qquad \blacksquare$$

THEOREM 3.6. *Let $(R, m)$ be a regular local ring an infinite residue field and $I$ an equimultiple ideal with $ht(I) = s > 0$. Then the following conditions are equivalent.*

*(1) $R[It]$ is a Cohen-Macaulay ring.*

*(2) $gr_I(R)$ is a Cohen-Macaulay ring.*

*Proof.* : (1) $\Longrightarrow$ (2) : This follows from Proposition 1.1 in [9].
(2) $\Longrightarrow$ (1) : By Lemma 3.5, there exist elements $a_1, \cdots, a_s$ in $I$ such that $I^s = (a_1, \cdots, a_s)I^{s-1}$. This implies $r(I) \leq s - 1$, which proves the assertion from Theorem 1.2. $\blacksquare$

COROLLARY 3.6.1. *(Huneke, [8], Proposition 2.6) Let $(R, m)$ be a regular local ring $\dim(R) = d > 0$ with an infinite residue field and $I$ an $m$-primary ideal of $R$. Then $R[It]$ is Cohen-Macaulay if and only if $gr_I(R)$ is Cohen-Macaulay.*

COROLLARY 3.6.2. *Let $(R, m)$ be a regular local ring and $I$ an ideal of $R$ with $ht(I) > 0$. Assume that $R$ is normally Cohen-Macaulay along $I$. Then*

$$depth(R[It]) = depth(gr_I(R)) + 1.$$

*Proof.* : Case (1) : If $gr_I(R)$ is not Cohen-Macaulay, then we have the equality by Theorem 3.4.
Case (2) : If $gr_I(R)$ is Cohen-Macaulay, then we see that $I$ is equimultiple since $R$ is normally Cohen-Macaulay along $I$. Hence we have the equality by Theorem 3.6. ∎

COROLLARY 3.6.3. *Let $(R, m)$ be a regular local ring of dimension $d > 0$ and $I$ an $m$-primary ideal. Then*

$$depth(R[It]) = depth(gr_I(R)) + 1.$$

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